STRATIFIED KÄHLER STRUCTURES ON ADJOINT QUOTIENTS

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Abstract. Given a compact Lie group, endowed with a bi-invariant Riemannian metric, its complexification inherits a Kähler structure having twice the kinetic energy of the metric as its potential, and Kähler reduction with reference to the adjoint action yields a stratified Kähler structure on the resulting adjoint quotient. Exploiting classical invariant theory, in particular bisymmetric functions and variants thereof, we explore the singular Poisson-Kähler geometry of this quotient. Among other things we prove that, for various compact groups, the real coordinate ring of the adjoint quotient is generated, as a Poisson algebra, by the real and imaginary parts of the fundamental characters. We also show that singular Kähler quantization of the geodesic flow on the reduced level yields the irreducible algebraic characters of the complexified group.

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Introduction

Given a smooth manifold M endowed with an action of a compact Lie group K, the action lifts to a hamiltonian action on the (total space of the) cotangent bundle T^*M in an obvious fashion. The reduced space at zero momentum is then a stratified symplectic space. In recent years various attempts have been made to understand the singular structure of this kind of reduced space; for example, unless there is a single stratum, the strata are not cotangent bundles on strata of the orbit space of M. In this paper we will elucidate the singular structure explicitly for the special case where M is K itself, endowed with the *conjugation* action. A choice of bi-invariant Riemannian metric amounts to fixing the kinetic energy, and we will show that quantization of the reduced kinetic energy on the resulting singular quotient yields the irreducible algebraic characters of the complexified group $K^{\mathbb{C}}$. Reduced spaces of this kind arise in mechanics, and the total space of the cotangent bundle on a compact Lie group with symmetry coming from conjugation is the building block for certain lattice gauge theories. Despite the huge literature on reduction of the cotangent bundle of a Lie group relative to *left translation* (or right translation), the *conjugation* action has received little attention. Quantization on the symplectic quotient (reduced space) of a space of the kind T^*K at zero momentum relative to the conjugation action will provide a step towards understanding quantization of certain constrained systems.

The polar decomposition of the complexification $K^{\mathbb{C}}$ of K and a choice of bi-invariant Riemannian metric on K induce a bi-invariant diffeomorphism between $K^{\mathbb{C}}$ and T^*K in such a way that the symplectic and complex structures combine to a Kähler structure. Then the reduced space at zero momentum $(\mathrm{T}^*K)_0$ may be identified with the complex algebraic categorical quotient $K^{\mathbb{C}}//K^{\mathbb{C}}$ and thereby acquires a complex algebraic structure which, when K is simple and simply connected of rank r (say), comes down to ordinary r-dimensional complex affine space. Indeed, in an obvious manner, the complex algebraic categorical quotient of $K^{\mathbb{C}}$ is isomorphic to the orbit space $T^{\mathbb{C}}\big/W$ of the complexification $T^{\mathbb{C}}$ of a maximal torus T in K relative to the action of the Weyl group W on $T^{\mathbb{C}}$. In the literature, an orbit space of the kind $T^{\mathbb{C}}\big/W$ is referred to as an *adjoint quotient*. When K is simple and simply connected of rank r (say), in view of an observation of *Steinberg*'s [39], the fundamental characters $\chi_1, \ldots, \chi_r$ of $K^{\mathbb{C}}$ furnish a map from $K^{\mathbb{C}}$ onto r-dimensional complex affine space $\mathbb{A}^r$ which identifies the adjoint quotient with $\mathbb{A}^r$. In particular, the complex coordinate ring $\mathbb{C}[T^{\mathbb{C}}\big/W]$ is the polynomial algebra in the characters of the fundamental irreducible representations, and this ring is that of W-invariants of the complex coordinate ring $\mathbb{C}[T^{\mathbb{C}}]$ of the maximal torus $[T^{\mathbb{C}}$ of $K^{\mathbb{C}}$.

However, the reduced space has more structure: We recall that a *complex analytic stratified Kähler space* in the sense of [17] is a stratified symplectic space $(N, C^\infty(N), \{\,\cdot\,,\,\cdot\,\})$ together with a compatible complex analytic structure on N. That is to say: N comes with (i) a stratification, with (ii) a Poisson algebra $(C^\infty(N), \{\,\cdot\,,\,\cdot\,\})$ of continuous functions, referred to as a *stratified symplectic Poisson algebra* which, on each stratum, restricts to an ordinary smooth symplectic Poisson algebra, and (iii) with a complex analytic structure, and the two structures being compatible amounts to the following additional requirements being satisfied: (iv) each stratum is a complex analytic subspace which is actually a complex manifold; (v) holomorphic functions, defined on open subsets of N, are restrictions of functions

in $C^\infty(N,\mathbb{C}) = C^\infty(N)\otimes\mathbb{C}$; and (vi) on each stratum, the symplectic and complex analytic structures combine to a Kähler structure. Given the compact Lie group K, the quotient $(\mathrm{T}^*K)_0$ is a complex analytic stratified Kähler space, indeed, even a "complex algebraic" stratified Kähler space in a sense made precise in Section 1 below. Thus, symplectically or, more precisely, as a stratified symplectic space, the quotient $(\mathrm{T}^*K)_0$ has singularities, even when K is simple and simply connected so that complex analytically or complex algebraically $(\mathrm{T}^*K)_0$ is just an affine space. Whether or not K is simply connected, Poisson brackets among the real and imaginary parts of holomorphic coordinate functions then yield (continuous) functions which are not necessarily smooth functions of the coordinate functions; indeed, the singular structure on the reduced level is reflected in Poisson brackets among (continuous) functions which are not necessarily smooth. Explicit examples of such Poisson brackets will be given in Sections 3 and 4 below. Suffice it to mention at this stage that the real coordinate ring $\mathbb{R}[T^{\mathbb{C}}/W]$ of the quotient $T^{\mathbb{C}}/W$ amounts to the algebra $\mathbb{R}[T^{\mathbb{C}}]^W$ of W-invariants in the real coordinate ring $\mathbb{R}[T^{\mathbb{C}}]$ of the maximal torus $T^{\mathbb{C}}$, viewed as a real algebraic manifold. Now the real coordinate ring $\mathbb{R}[T^{\mathbb{C}}/W]$ of the quotient $T^{\mathbb{C}}/W$ contains of course the subalgebra generated by the real and imaginary parts of the polynomial generators of $\mathbb{C}[T^{\mathbb{C}}/W]$ but these do not generate the real coordinate ring of the quotient. The stratified symplectic Poisson structure of the quotient is defined on $\mathbb{R}[T^{\mathbb{C}}/W]$, and one of our results is the following.

Theorem. *For $K = \mathrm{U}(n), \mathrm{SU}(n), \mathrm{Sp}(n), \mathrm{SO}(2n+1,\mathbb{R}), G_{2(-14)}$, as a Poisson algebra, the real coordinate ring $\mathbb{R}[T^{\mathbb{C}}/W]$ of the quotient $T^{\mathbb{C}}/W$ is generated by the real and imaginary parts of the characters $\chi_1,\ldots,\chi_r$ of the fundamental irreducible representations of $K^{\mathbb{C}}$. That is to say: This ring is generated by the real and imaginary parts of these characters, together with iterated Poisson brackets in these functions.*

For $K = \mathrm{U}(n)$ or $K = \mathrm{SU}(n)$, the theorem comes down to the statement that the algebra $\mathbb{C}[z_1,\ldots,z_n,\overline{z}_1,\ldots,\overline{z}_n]^{S_n}$ of bisymmetric functions, that is, the algebra of S_n-invariants in the variables $z_1,\ldots,z_n,\overline{z}_1,\ldots,\overline{z}_n$, where the symmetric group S_n permutes the variables $z_1,\ldots,z_n$ and $\overline{z}_1,\ldots,\overline{z}_n$ separately, is generated by the elementary symmetric functions $\sigma_1,\ldots,\sigma_n$ in the variables $z_1,\ldots,z_n$ and the elementary symmetric functions $\overline{\sigma}_1,\ldots,\overline{\sigma}_n$ in the variables $\overline{z}_1,\ldots,\overline{z}_n$, together with iterated Poisson brackets in these functions. See Corollaries 3.4.4 and 3.4.8 below for more details. The cases $\mathrm{Sp}(n)$, $\mathrm{SO}(2n+1,\mathbb{R})$, and $G_{2(-14)}$ are established in Section 4.

For $K = \mathrm{Spin}(n,\mathbb{R})$ $(n\geq 3,\ n\neq 4,\ n\neq 6)$, $K = \mathrm{SO}(2n,\mathbb{R})$ $(n\geq 2)$, $K = F_{2(-52)}$, the statement of the theorem is not true. We do not know what happens for $K = E_{6(-78)}, E_{7(-132)}, E_{8(-248)}$.

For $K = \mathrm{U}(n)$, the unitary group, complex algebraically, the reduced space $(\mathrm{T}^*K)_0 \cong \mathrm{GL}(n,\mathbb{C})//\mathrm{GL}(n,\mathbb{C})$ is the space of complex normalized degree n polynomials in a single variable having non-zero constant coefficient, and this space is complex algebraically and hence complex analytically isomorphic to $\mathbb{C}^{n-1}\times\mathbb{C}^*$ in an obvious fashion. Indeed, the quotient map from $K^{\mathbb{C}} \cong \mathrm{GL}(n,\mathbb{C})$ to the space of polynomials sends a matrix in $\mathrm{GL}(n,\mathbb{C})$ to its characteristic polynomial, and the stratification of the reduced space is given by the multiplicities of the roots where strata correspond to partitions of n. The stratified symplectic Poisson al-

gebra $C^\infty(\mathrm{T}^*K)_0$ contains more functions than the ordinary smooth functions on $(\mathrm{T}^*K)_0 = \mathbb{C}^{n-1} \times \mathbb{C}^*$, though, viewed as a smooth real $(2n)$-dimensional manifold. In Section 3 below shall determine the stratified symplectic Poisson structure explicitly. For $K = \mathrm{SU}(n)$, the quotient $\mathrm{SL}(n,\mathbb{C})//\mathrm{SL}(n,\mathbb{C})$ amounts to the subspace of the quotient $\mathrm{GL}(n,\mathbb{C})//\mathrm{GL}(n,\mathbb{C})$ for $\mathrm{U}(n)$ which consists of complex normalized degree n polynomials with constant coefficient equal to 1. Complex algebraically, this space is plainly just a copy of $\mathbb{C}^{n-1}$.

In the final section we shall show that half-form quantization of the reduced kinetic energy associated with the bi-invariant metric on K yields the irreducible algebraic characters of $K^{\mathbb{C}}$. In the situation considered there, quantization unitarily commutes with reduction. Exploiting results in [34], we plan to extend elsewhere the present approach to orbit spaces of n-tuples of elements from T^*K. This is the typical situation in lattice gauge theory.

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1. The adjoint quotient

Let K be a compact Lie group, let $\mathfrak{k}$ be its Lie algebra, choose an invariant inner product $\cdot\colon \mathfrak{k}\otimes\mathfrak{k} \to \mathbb{R}$ on $\mathfrak{k}$, and endow K with the corresponding bi-invariant Riemannian metric. Using this metric, we identify $\mathfrak{k}$ with its dual $\mathfrak{k}^*$ and the total space of the tangent bundle $\mathrm{T}K$ with the total space of the cotangent bundle T^*K. The polar decomposition map assigns $x\cdot\exp(iY) \in K^{\mathbb{C}}$ to $(x, Y) \in K \times \mathfrak{k}$. Thus the composite

$$\mathrm{T}^*K \longrightarrow K \times \mathfrak{k} \longrightarrow K^{\mathbb{C}} \tag{1.1}$$

of the inverse of left trivialization with the polar decomposition map identifies T^*K with $K^{\mathbb{C}}$ in a $(K \times K)$-equivariant fashion. Then the induced complex structure on T^*K combines with the symplectic structure to a (positive) Kähler structure. Indeed, the *real analytic* function

$$\kappa\colon K^{\mathbb{C}} \longrightarrow \mathbb{R}, \quad \kappa(x\cdot\exp(iY)) = |Y|^2, \quad (x, Y) \in K \times \mathfrak{k}, \tag{1.2}$$

on $K^{\mathbb{C}}$ which is twice the *kinetic energy* associated with the Riemannian metric, is a (globally defined) *Kähler potential*; in other words, the function κ is strictly plurisubharmonic and (the negative of the imaginary part of) its Levi form yields (what corresponds to) the cotangent bundle symplectic structure, that is, the cotangent bundle symplectic structure on T^*K is given by $i\partial\overline{\partial}\kappa$. An explicit calculation which establishes this fact may be found in [13] (but presumably it is a folk-lore observation). For related questions see [27], [40].

The group K acts on itself and hence on the total space T^*K of its cotangent bundle via conjugation. The K-action on $\mathrm{T}^*K \cong \mathrm{T}K$ is hamiltonian and preserves the Kähler structure, with momentum mapping

$$\mu\colon \mathrm{T}K \to \mathfrak{k}, \quad \mu(X_x) = X_x x^{-1} - x^{-1}X_x, \tag{1.3}$$

where $x \in K$, where $X_x \in \mathrm{T}_xK$ is a tangent vector at x, and where $X_x x^{-1} \in \mathfrak{k}$ and $x^{-1}X_x \in \mathfrak{k}$ are the results of right and left translation, respectively, with x^{-1}. By Proposition 4.2 of [17], the Kähler quotient $\mathrm{T}^*K//K$ at zero momentum is a complex analytic stratified Kähler space.

Symplectically, the quotient is the orbit space $\mu^{-1}(0)/K$, and an observation of Kempf-Ness [25] and Kirwan [26], cf. §4 of [36], where the zero locus $\mu^{-1}(0)$ is referred to as a *Kempf-Ness* set, entails that the obvious map

$$\mu^{-1}(0)/K \to \mathrm{T}^*K//K^{\mathbb{C}} \cong K^{\mathbb{C}}//K^{\mathbb{C}}. \tag{1.4}$$

from the symplectic quotient (reduced space) to the categorical quotient induced by the inclusion of $\mu^{-1}(0)$ into T^*K is a homeomorphism in the ordinary (not Zariski) topology. Here $K^{\mathbb{C}}//K^{\mathbb{C}}$ refers to the complex algebraic categorical quotient of $K^{\mathbb{C}}$ relative to the $K^{\mathbb{C}}$-action on itself via conjugation; see e. g. [36] (§3) for details on the construction of the categorical quotient in the category of complex algebraic varieties. In view of results of Luna [29], [30], this quotient is the categorical quotient in the category of analytic varieties as well, see also [35] (Theorem 3.6).

The categorical quotient $K^{\mathbb{C}}//K^{\mathbb{C}}$, in turn, has a very simple structure: Choose a maximal torus T in K and let W be the corresponding Weyl group; then $T^{\mathbb{C}}$ is a maximal torus in $K^{\mathbb{C}}$, and the (algebraic) *adjoint quotient* $\chi\colon K^{\mathbb{C}} \to T^{\mathbb{C}}/W$, cf. [24] (3.4) and [37] (3.2) for this terminology, realizes the categorical quotient. Here $T^{\mathbb{C}}/W$ is the space of W-orbits in $T^{\mathbb{C}}$, and we will also refer to the orbit space $T^{\mathbb{C}}/W$ as the *adjoint quotient of* $K^{\mathbb{C}}$. In concrete terms, the map χ admits the following description: The closure of the conjugacy class of $x \in K^{\mathbb{C}}$ contains a unique semisimple (equivalently: closed) conjugacy class C_x (say), and semisimple conjugacy classes are parametrized by $T^{\mathbb{C}}/W$; the image of $x \in K^{\mathbb{C}}$ under χ is simply the parameter value in $T^{\mathbb{C}}/W$ of the semisimple conjugacy class C_x. Since W is a finite group, as a complex algebraic space, the quotient $T^{\mathbb{C}}/W$ is simply the space of W-orbits in $T^{\mathbb{C}}$.

The choice of maximal torus T in K also provides considerable simplification for the stratified symplectic structure on the symplectic quotient $(\mathrm{T}^*K)_0 = \mu^{-1}(0)/K$. Indeed, via the identification (1.1) for $K = T$, the real space which underlies the (complex algebraic) orbit space $T^{\mathbb{C}}/W$ for the action of the Weyl group W on $T^{\mathbb{C}}$ amounts simply to the orbit space T^*T/W, with reference to the induced action of the Weyl group W on T^*T, and the orbit space T^*T/W inherits a stratified symplectic structure in an obvious fashion: Strata are the W-orbits, the closures of the strata are affine varieties, and the requisite stratified symplectic Poisson algebra $(C^\infty(\mathrm{T}^*T/W), \{\,\cdot\,,\cdot\,\})$ is simply the algebra $C^\infty(\mathrm{T}^*T)^W$ of smooth W-invariant functions on T^*T, endowed with the Poisson bracket $\{\,\cdot\,,\cdot\,\}$ coming from the ordinary symplectic Poisson bracket on T^*T. Moreover, the choice of invariant inner product on $\mathfrak{k}$ determines an injection $\mathrm{T}^*T \to \mathrm{T}^*K$.

Proposition 1.5. *The values of the injection* $\mathrm{T}^*T \to \mathrm{T}^*K$ *lie in the zero locus* $\mu^{-1}(0)$*, and the injection* $\mathrm{T}^*T \to \mu^{-1}(0)$ *induces an isomorphism*

$$\mathrm{T}^*T/W \longrightarrow (\mathrm{T}^*K)_0 = \mu^{-1}(0)/K$$

of stratified symplectic spaces.

Proof. Since T is an abelian Lie group, and since the action of K on T^*K is by conjugation, the restriction of the momentum mapping μ to T^*T, cf. (1.3) above, is zero, that is, the values of the injection $\mathrm{T}^*T \to \mathrm{T}^*K$ lie in the zero locus $\mu^{-1}(0)$. Since the map (1.4) and the obvious map from $T^{\mathbb{C}}/W$ to $K^{\mathbb{C}}//K^{\mathbb{C}}$ are homeomorphisms, the induced map from T^*T/W to $(\mathrm{T}^*K)_0 = \mu^{-1}(0)/K$ is a homeomorphism as well. Moreover, under this map, the orbit type stratifications correspond.

Any smooth K-invariant function on T^*K restricts to a smooth W-invariant function on T^*T. Consequently the homeomorphism from T^*T/W to $(\mathrm{T}^*K)_0 = \mu^{-1}(0)/K$ induces a map

$$C^\infty((\mathrm{T}^*K)_0) = C^\infty(\mathrm{T}^*K)^K/I^K \longrightarrow (C^\infty(\mathrm{T}^*T))^W,$$

necessarily injective since the map between the underlying spaces is a homeomorphism. Thus is remains to show that each smooth W-invariant function on T^*T extends to a smooth K-invariant function on T^*K, with reference to the conjugation action on T^*K. However, this follows from the compactness of K: A smooth W-invariant function f on T^*T extends to a smooth function $\widetilde{F}$ on T^*K since T^*T is a closed submanifold of T^*T; averaging over K then yields a smooth K-invariant function F on T^*K extending f. $\square$

Thus the *real* structure $C^\infty((\mathrm{T}^*K)_0)$ comes down to the algebra $C^\infty(\mathrm{T}^*T)^W$ of (real) smooth functions on $\mathrm{T}^*T \cong T^{\mathbb{C}}$ that are invariant under the action of the Weyl group W.

The quotient $(\mathrm{T}^*K)_0 \cong T^{\mathbb{C}}/W$ inherits various interrelated structures, and for later reference we will now spell them out and introduce appropriate notation: Given a real affine locally semialgebraic space N (embedded into some some real affine space), we write its real coordinate ring as $\mathbb{R}[N]$ and, accordingly, we write its ring of analytic functions and that of Whitney smooth functions as $C^\omega(N)$ and $C^\infty(N)$, respectively. Likewise, given an affine complex variety N, we denote its complex coordinate ring by $\mathbb{C}[N]$. By construction, on the adjoint quotient $N = T^{\mathbb{C}}/W$, the algebra $\mathbb{R}[N] = \mathbb{R}[T^{\mathbb{C}}]^W$ of W-invariant real polynomial functions on $T^{\mathbb{C}}$ yields a real affine locally semialgebraic structure, the algebra $\mathbb{C}[N] = \mathbb{C}[T^{\mathbb{C}}]^W$ of W-invariant complex polynomial functions on $T^{\mathbb{C}}$ yields a complex affine structure, and the real Poisson structure is real algebraic in the sense that it is defined already on $\mathbb{R}[N]$. Further, these structures combine to a *complex algebraic stratified Kähler structure* on N, that is, the Poisson structure is already defined on $\mathbb{R}[N]$ and the complex structure is given in terms of the complex affine coordinate ring $\mathbb{C}[N]$ but, beware, the algebra $\mathbb{C}[N]$ is *not* the complexification of $\mathbb{R}[N]$. Indeed, $\mathbb{C}[N]$ may be identified with a subalgebra of the complexification $\mathbb{R}[N]_{\mathbb{C}}$ of $\mathbb{R}[N]$ but $\mathbb{R}[N]_{\mathbb{C}}$ is strictly larger than $\mathbb{C}[N]$; see Section 3 below for concrete examples. Moreover, the algebra $C^\omega(N) = C^\omega(T^{\mathbb{C}})^W$ of W-invariant real analytic functions on $T^{\mathbb{C}}$ yields a real affine

locally semianalytic structure on N, and the three real structures are related by the obvious embeddings

$$\mathbb{R}[N] \subseteq C^\omega(N) \subseteq C^\infty(N).$$

In particular, the Kähler potential κ (twice the kinetic energy) on $T^{\mathbb{C}}$ is a real analytic function which is plainly W-invariant and hence descends to a function κ_{red} in $C^\omega(N)$, the *reduced* Kähler potential, which is then twice the reduced kinetic energy. This function is a Kähler potential on the adjoint quotient N in the sense that, restricted to a stratum, it yields an ordinary Kähler potential on that stratum. Notice that κ_{red} does not belong to the real coordinate ring $\mathbb{R}[N]$ of the adjoint quotient N, though, and the reduced Kähler potential is not an ordinary smooth function, that is, it is neither a real analytic nor a smooth function on the adjoint quotient, even when this quotient is topologically just an affine space. A description of the Poisson bracket on the real coordinate ring $\mathbb{R}[N]$ of the adjoint quotient N will be given in the next section.

It is, perhaps, worthwhile pointing out that, on $T^{\mathbb{C}}$ and, likewise, on $K^{\mathbb{C}}$, the Kähler structure is algebraic while the Kähler potential is a real analytic function. On the other hand, the total spaces $\mathrm{T}T$ and $\mathrm{T}K$ of the tangent bundles of the maximal torus T of K and of K itself, respectively, are complex analytically equivalent to $T^{\mathbb{C}}$ and $K^{\mathbb{C}}$, respectively, under the polar map, the Poisson structures on $\mathrm{T}T \cong \mathrm{T}^*T$ and on $\mathrm{T}K \cong \mathrm{T}^*K$ are real algebraic (even though the identification between T^*K and $K^{\mathbb{C}}$ is real analytic), the induced complex structures on $\mathrm{T}T$ and on $\mathrm{T}K$ are real analytic and, on T^*T and T^*K, the Kähler potentials are real algebraic functions. Thus we could describe the stratified Kähler structure on the symplectic quotient $(\mathrm{T}^*K)_0$ in terms of the Kähler structure on T^*T as well but this would yield a stratified Kähler structure which is just complex analytic and *not* algebraic. On the other hand, the two stratified Kähler structures are plainly equivalent in the category of complex analytic stratified Kähler spaces.

2. The reduced Poisson algebra

As before, let K be a compact Lie group, let T be a maximal torus of K, let $n = \dim(T) = \mathrm{rank}(K)$, and let W be the corresponding Weyl group. In view of Proposition 1.5, as a stratified symplectic space, the symplectic quotient $(\mathrm{T}^*K)_0 = \mu^{-1}(0)\big/K$ amounts to the orbit space $\mathrm{T}^*T\big/W$ and, since W acts symplectically, the Poisson algebra of smooth W-invariant functions on $\mathrm{T}^*T \cong T^{\mathbb{C}}$ yields the *reduced Poisson algebra*, that is, the stratified symplectic Poisson algebra on the quotient. The purpose of the present section is to derive an explicit description of this reduced Poisson algebra. Actually, this Poisson algebra is real algebraic, and we will describe it as a real algebraic object. To this end we realize the W-manifold $T^{\mathbb{C}} \cong (\mathbb{C}^*)^n$ as a closed non-singular W-variety in a suitable W-representation. We examine first a special case, in a manner which may look unnecessarily complicated but which will pave the way towards quantization on the adjoint quotient in Section 5 below.

(2.1) $K = \mathrm{U}(1) = S^1$, THE CIRCLE GROUP. We identify the Lie algebra $\mathrm{Lie}(S^1)$ of the circle group S^1 with the real numbers $\mathbb{R}$ by means of the embedding $\mathbb{R} \to \mathbb{C}$ given by the association $s \mapsto -is$, and we endow S^1 with the standard Riemannian metric. The complexification of the circle group amounts to a copy of $\mathbb{C}^*$. We identify the Lie algebra $\mathrm{Lie}(\mathbb{C}^*)$ of $\mathbb{C}^*$ with a copy of the complex numbers $\mathbb{C}$, and we will use the holomorphic coordinate $w = t + is$ on this Lie algebra. The exponential mapping

from $\mathbb{C}$ to $\mathbb{C}^*$ factors as

$$\exp\colon \mathbb{C} \longrightarrow S^1 \times \mathbb{R} \longrightarrow \mathbb{C}^*$$

where the first arrow is the universal covering projection (which sends $w = t + is$ to (e^{is}, t)) and the second one the polar map

$$S^1 \times \mathbb{R} \cong \mathrm{T}S^1 \longrightarrow \mathbb{C}^*, \quad (\mathrm{e}^{is}, t) \mapsto \mathrm{e}^t \mathrm{e}^{is},\ s, t \in \mathbb{R}. \tag{2.1.1}$$

The Kähler potential (1.2) on $S^1 \times \mathbb{R}$, combined with the universal covering projection from $\mathbb{C}$ to $S^1 \times \mathbb{R}$, yields the Kähler potential $\widetilde{\kappa}$ on $\mathbb{C}$ given by

$$\widetilde{\kappa}(w) = t^2 = \left(\frac{w + \overline{w}}{2}\right)^2 = \frac{1}{2} w\overline{w} + \frac{1}{4} w^2 + \frac{1}{4} \overline{w}^2; \tag{2.1.2}$$

this Kähler potential yields the standard Kähler form on $\mathbb{C}$ and is manifestly invariant under the group of deck transformations. Notice that the standard Kähler potential on $\mathbb{C}$ (given by the assignment to $w \in \mathbb{C}$ of $\frac{w\overline{w}}{2}$) is *not* invariant under the group of deck transformations. It follows that the induced Kähler structure on $\mathrm{T}S^1$ is the ordinary flat one, and the universal covering projection from $\mathfrak{k}^{\mathbb{C}} = \mathbb{C}$ to $\mathrm{T}S^1$ is compatible with the Kähler structures where the universal covering space $\mathbb{C}$ carries the standard structure. To arrive at an explicit formula on $\mathrm{T}S^1 \cong S^1 \times \mathbb{R}$, write the Maurer-Cartan form on S^1 as ds and let $z = x + iy$ be the holomorphic coordinate on (the copy of) $\mathbb{C}$ (downstairs), so that $\mathbb{C}^* \cong \mathrm{T}S^1$ appears as a subspace of $\mathbb{C}$; thus, in terms of the coordinates t and s appearing in (2.1.1), $x = \mathrm{e}^t \cos(s)$ and $y = \mathrm{e}^t \sin(s)$. Under the polar map, the cotangent bundle symplectic structure $dtds$ on $\mathrm{T}S^1 \cong \mathrm{T}^*S^1$ (strictly speaking this is the negative of the ordinary cotangent bundle symplectic structure) and the standard symplectic structure $dxdy$ on $\mathbb{C}^*$ (viewed as a subspace of $\mathbb{C}$) are related by $dxdy = \mathrm{e}^{2t} dtds$ whence the induced symplectic structure on $\mathbb{C}^*$ is given by $\frac{1}{r^2} dxdy$ where $r^2 = z\overline{z}$. Indeed, on $\mathbb{C}^*$, since

$$t = \frac{1}{2} \log(z\overline{z}) = \frac{1}{2} \log r^2 = \log r,$$

the Kähler potential κ (cf. (1.2)) is given by

$$\kappa(z) = t^2 = \log^2(\sqrt{z\overline{z}}) = \frac{1}{4} \log^2(z\overline{z}) \tag{2.1.3}$$

where $\log^2(v) = (\log v)^2$, whence

$$i\partial\overline{\partial}\kappa = \frac{i}{2r^2} dz \wedge d\overline{z} = \frac{1}{r^2} dx \wedge dy. \tag{2.1.4}$$

Consequently, in terms of the variables x and y, the Poisson bracket on the real coordinate ring $\mathbb{R}[\mathbb{C}^*]$ is given by

$$\{x, y\} = r^2 \tag{2.1.5}$$

where $r^2 = x^2 + y^2$ as usual; this yields a Poisson structure on the algebra of smooth real functions $C^\infty(\mathbb{C}^*)$ on $\mathbb{C}^*$ in the standard fashion. Alternatively, in terms of the

variables z and $\overline{z}$, the Poisson bracket on the complexification $\mathbb{R}[\mathbb{C}^*]_{\mathbb{C}}$ of the real coordinate ring $\mathbb{R}[\mathbb{C}^*]$ is given by

$$\frac{i}{2}\{z,\overline{z}\} = z\overline{z}, \tag{2.1.6}$$

and this yields a Poisson structure on the algebra of smooth complex functions $C^\infty(\mathbb{C}^*,\mathbb{C})$ on $\mathbb{C}^*$.

(2.2) The case of a general compact connected Lie group K: Let T be a maximal torus of K, let $n = \dim T = \mathrm{rank}(K)$, and let W be the Weyl group.

(2.2.1) The complex structure. As a Kähler manifold, the complex torus $T^{\mathbb{C}}$ comes down to a product $(\mathbb{C}^*)^n$ of n copies of $\mathbb{C}^*$, each copy of $\mathbb{C}^*$ being endowed with the symplectic structure corresponding to the Poisson structure (2.1.5). The W-action on $T^{\mathbb{C}}$ induces a W-module structure on the complex affine coordinate ring $\mathbb{C}[T^{\mathbb{C}}]$ in an obvious fashion. Choose coordinate functions $v_1,\dots,v_m$ in $\mathbb{C}[T^{\mathbb{C}}]$ which generate $\mathbb{C}[T^{\mathbb{C}}]$ as an algebra and such that the $\mathbb{C}$-linear span of $v_1,\dots,v_m$ in $\mathbb{C}[T^{\mathbb{C}}]$ is W-invariant. The assignment to a point q of $T^{\mathbb{C}}$ of

$$(v_1(q),\dots,v_m(q)) \in V = \mathbb{C}^m$$

yields an embedding

$$T^{\mathbb{C}} \cong (\mathbb{C}^*)^n \to V = \mathbb{C}^m \tag{2.2.1.1}$$

of $T^{\mathbb{C}} \cong (\mathbb{C}^*)^n$ into $V = \mathbb{C}^m$ as a closed non-singular complex subvariety of V. By construction, V is endowed with a W-module structure, and the embedding (2.2.1.1) is W-equivariant. Let $z_1,\dots,z_m$ be the obvious coordinates on V. The corresponding morphism

$$\mathbb{C}[V] = \mathbb{C}[z_1,\dots,z_m] \to \mathbb{C}[T^{\mathbb{C}}] \tag{2.2.1.2}$$

of $\mathbb{C}$-algebras is given by the assignment to z_j of v_j $(1 \le j \le m)$ and realizes the complex affine coordinate ring $\mathbb{C}[T^{\mathbb{C}}]$ as the quotient of $\mathbb{C}[V]$ given by suitable relations. Moreover the $\mathbb{C}$-algebra $\mathbb{C}[V]$ inherits a W-module structure in an obvious fashion, and the morphism (2.2.1.2) of $\mathbb{C}$-algebras is W-equivariant. By a theorem of Hilbert, the algebra of W-invariants $\mathbb{C}[V]^W$ is finitely generated. This algebra is the complex coordinate ring $\mathbb{C}[V/W]$ of the quotient V/W, viewed as a complex affine variety. Since W is finite, the induced map

$$\mathbb{C}[V]^W \to \mathbb{C}[T^{\mathbb{C}}]^W \tag{2.2.1.3}$$

to the algebra $\mathbb{C}[T^{\mathbb{C}}]^W$ of W-invariants in $\mathbb{C}[T^{\mathbb{C}}]$ is surjective. This algebra of invariants is the complex coordinate ring $\mathbb{C}[T^{\mathbb{C}}/W]$ of the quotient $T^{\mathbb{C}}/W$, and the surjection (2.2.1.3) is dual to the induced embedding of $T^{\mathbb{C}}/W$ into V/W. Thus a choice $f_1,\dots,f_k$ of multiplicative generators of $\mathbb{C}[V]^W$ induces embeddings $T^{\mathbb{C}}/W \subseteq V/W \subseteq \mathbb{C}^k$ which realize $T^{\mathbb{C}}/W$ and V/W as complex affine varieties in $\mathbb{C}^k$.

(2.2.2) The real semialgebraic structure. For $1 \le j \le m$, let $z_j = x_j + y_j$, let $\mathbb{R}[V] = \mathbb{R}[x_1,y_1,\dots,x_m,y_m]$, the real coordinate ring of V where V is viewed as a real $(2m)$-dimensional vector space and, $T^{\mathbb{C}}$ being viewed as a real $(2n)$-dimensional

non-singular variety, let $\mathbb{R}[T^{\mathbb{C}}]$ denote its real coordinate ring. By construction, $\mathbb{R}[T^{\mathbb{C}}]$ is multiplicatively generated by $\mathrm{Re}(v_1), \mathrm{Im}(v_1), \ldots, \mathrm{Re}(v_m), \mathrm{Im}(v_m)$, and the assignment to x_j and y_j of $\mathrm{Re}(v_j)$ and $\mathrm{Im}(v_j)$, respectively, $(1 \leq j \leq m)$ yields a W-equivariant surjection from $\mathbb{R}[V]$ onto $\mathbb{R}[T^{\mathbb{C}}]$ which is dual to the W-equivariant embedding (2.2.1.1) of $T^{\mathbb{C}}$ into V, both spaces being viewed as real (non-singular affine) W-varieties. By the theorem of Hilbert quoted earlier, the algebra of real W-invariants $\mathbb{R}[V]^W$ is finitely generated. This algebra is the *real* coordinate ring $\mathbb{R}[V/W]$ of the quotient V/W, viewed as a real semialgebraic space (see below). Since W is finite, the induced map

$$\mathbb{R}[V]^W \longrightarrow \mathbb{R}[T^{\mathbb{C}}]^W \tag{2.2.2.1}$$

to the algebra $\mathbb{R}[T^{\mathbb{C}}]^W$ of W-invariants in $\mathbb{R}[T^{\mathbb{C}}]$ is surjective. This algebra of invariants is the *real* coordinate ring $\mathbb{R}[T^{\mathbb{C}}/W]$ of the quotient $T^{\mathbb{C}}/W$, and the surjection (2.2.2.1) is dual to the induced embedding of $T^{\mathbb{C}}/W$ into V/W, both spaces being viewed as real semialgebraic sets. These quotients are not ordinary real varieties, that is, neither of them is the space of real points of a complex variety. To spell out the structure somewhat more explicitly, pick a finite system of generators $\alpha_1, \ldots, \alpha_\ell$ of $\mathbb{R}[V]^W$ and let

$$\alpha = (\alpha_1, \ldots, \alpha_\ell) \colon V \longrightarrow \mathbb{R}^\ell.$$

This map induces an embedding of the quotient V/W into $\mathbb{R}^\ell$ which realizes V/W as a real *semialgebraic* set in $\mathbb{R}^\ell$ [4] (§1). Defining relations for the algebra $\mathbb{R}[V]^W$ of W-invariants then yield defining equations for the smallest *real* variety $\widehat{V/W}$ in which V/W lies, that is, for the corresponding real *categorical* quotient $\widehat{V/W}$ of V; as a subspace of this categorical quotient, the quotient V/W is then given by a finite set of inequalities which, in turn, encapsulate the real semialgebraic structure of V/W. The real categorical quotient $\widehat{T^{\mathbb{C}}/W}$ of $T^{\mathbb{C}}$ and the quotient $T^{\mathbb{C}}/W$ which are our primary objects of interest are related in the same fashion: The image of $T^{\mathbb{C}}$ in V can be described by a single W-invariant equation (by suitable sums of squares if need be), and the categorical quotient $\widehat{T^{\mathbb{C}}/W}$ embeds into $\widehat{V/W}$ as a real algebraic subset of $\widehat{V/W}$. Hence $T^{\mathbb{C}}/W \cong (V/W \cap \widehat{T^{\mathbb{C}}/W}) \subseteq \widehat{V/W}$. Thus the inequalities determining V/W in $\widehat{V/W}$ determine $T^{\mathbb{C}}/W$ as a semialgebraic subset of $\widehat{T^{\mathbb{C}}/W}$. See e. g. [36] for more details.

(2.2.3) The real algebraic Poisson structure. Identify $T^{\mathbb{C}}$ with the product of n copies of $\mathbb{C}^*$, each copy of $\mathbb{C}^*$ being endowed with the Poisson structure (2.1.5); now, calculating the brackets among the images $\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_\ell \in \mathbb{R}[T^{\mathbb{C}}]^W$ of the generators $\alpha_1, \ldots, \alpha_\ell$ of the algebra $\mathbb{R}[V]^W$ of W-invariants, we obtain a description of the resulting Poisson structure on the real coordinate ring $\mathbb{R}[T^{\mathbb{C}}/W]$.

Alternatively, we may describe the resulting Poisson structure on $\mathbb{R}[T^{\mathbb{C}}/W]$ in terms of the complexification $\mathbb{R}[T^{\mathbb{C}}/W]_{\mathbb{C}}$. This provides considerable simplification, as we shall illustrate in Section 3 below. Indeed, the complexification $\mathbb{R}[V]_{\mathbb{C}}$ of $\mathbb{R}[V]$ (beware: this complexification is not the algebra $\mathbb{C}[V]$ considered above) is the polynomial algebra

$$\mathbb{R}[V]_{\mathbb{C}} = \mathbb{C}[z_1, \overline{z}_1, \ldots, z_m, \overline{z}_m] \tag{2.2.3.1}$$

in the variables $z_1, \overline{z}_1, \ldots, z_m, \overline{z}_m$, as a $\mathbb{C}$-algebra, the complexification $\mathbb{R}[T^{\mathbb{C}}]_{\mathbb{C}}$ of $\mathbb{R}[T^{\mathbb{C}}]$ is multiplicatively generated by $v_1, \overline{v}_1, \ldots, v_m, \overline{v}_m$, the algebras $\mathbb{R}[V]_{\mathbb{C}}$ and $\mathbb{R}[T^{\mathbb{C}}]_{\mathbb{C}}$ carry induced W-module structures in an obvious fashion, and the assignment to z_j of v_j and to $\overline{z}_j$ of $\overline{v}_j$ $(1 \leq j \leq m)$ yields the corresponding W-equivariant surjection from $\mathbb{R}[V]_{\mathbb{C}}$ onto $\mathbb{R}[T^{\mathbb{C}}]_{\mathbb{C}}$. Furthermore, the algebra of W-invariants $\mathbb{R}[V]_{\mathbb{C}}^W$ is still finitely generated. This algebra is the complexification of the *real* coordinate ring of the quotient V/W, viewed as a real semialgebraic space. Since W is finite, the induced map

$$\mathbb{R}[V]_{\mathbb{C}}^W = \mathbb{C}[z_1, \overline{z}_1, \ldots, z_m, \overline{z}_m]^W \longrightarrow \mathbb{R}[T^{\mathbb{C}}]_{\mathbb{C}}^W \tag{2.2.3.2}$$

to the algebra $\mathbb{R}[T^{\mathbb{C}}]_{\mathbb{C}}^W$ of W-invariants in $\mathbb{R}[T^{\mathbb{C}}]_{\mathbb{C}}$ is surjective. This algebra of invariants is the complexification $\mathbb{R}[T^{\mathbb{C}}/W]_{\mathbb{C}}$ of the real coordinate ring $\mathbb{R}[T^{\mathbb{C}}/W]$ of the quotient $T^{\mathbb{C}}/W$. Similarly as before, we may now pick a finite system of generators $\beta_1, \ldots, \beta_\ell$ of $\mathbb{R}[V]_{\mathbb{C}}^W = \mathbb{C}[z_1, \overline{z}_1, \ldots, z_m, \overline{z}_m]^W$. In particular, once a choice $f_1, \ldots, f_k$ of multiplicative generators of $\mathbb{C}[V]^W$, referred to henceforth as *complex* invariants, has been made, cf. what was said above, these complex generators and their complex conjugates yield $2k$ invariants in $\mathbb{C}[z_1, \overline{z}_1, \ldots, z_m, \overline{z}_m]^W$; however, these $2k$ invariants will not generate the algebra of invariants, and the system of invariants must be completed by *mixed* invariants, that is, by invariants involving the z_j's and the $\overline{z}_j$'s $(1 \leq j \leq k)$.

(2.2.4) THE SMOOTH POISSON STRUCTURE. In view of a result of G. W. SCHWARZ [35], every smooth W-invariant function of the variables $x_1, y_1, \ldots, x_m, y_m$ can be written as a smooth function of the variables $\alpha_1, \ldots, \alpha_\ell$, that is, under the projection from V to V/W, the algebra of Whitney-smooth functions on V/W (relative to the embedding into $\mathbb{R}^\ell$) is identified with the algebra $C^\infty(V)^W$ of smooth W-invariant functions on V. Furthermore, since W is finite, the induced map

$$C^\infty(V)^W \to C^\infty(T^{\mathbb{C}})^W$$

is surjective whence every smooth W-invariant function on $T^{\mathbb{C}}$ can be written as a smooth W-invariant function of the variables $x_1, y_1, \ldots, x_m, y_m$ and hence as a smooth function in the variables $\alpha_1, \ldots, \alpha_\ell$. Consequently, under the projection from $T^{\mathbb{C}}$ to $T^{\mathbb{C}}/W$, the algebra of real Whitney-smooth functions on $T^{\mathbb{C}}/W$ (relative to the embedding into $\mathbb{R}^\ell$) is identified with the algebra $C^\infty(T^{\mathbb{C}})^W$ of real smooth W-invariant functions on $T^{\mathbb{C}}$.

The Poisson structure on the real coordinate ring $\mathbb{R}[T^{\mathbb{C}}/W]$ induces the reduced Poisson structure $\{\,\cdot\,,\,\cdot\,\}$ on the algebra $C^\infty(N)$ in terms of generators of the algebra $C^\infty(T^{\mathbb{C}}/W) \cong C^\infty(T^{\mathbb{C}})^W$ as promised at the beginning of the present section. Likewise, in view of the quoted result of G. W. SCHWARZ, every smooth complex W-invariant function of the variables $z_1, \overline{z}_1, \ldots, z_m, \overline{z}_m$ can be written as a smooth function in the variables $\beta_1, \ldots, \beta_\ell$, and, likewise, every smooth complex W-invariant function on $T^{\mathbb{C}}$ can be written as a smooth W-invariant function of the variables $z_1, \overline{z}_1, \ldots, z_m, \overline{z}_m$ and hence as a smooth function in the variables $\beta_1, \ldots, \beta_\ell$. Now, with reference to the product Poisson bracket on $C^\infty(T^{\mathbb{C}}, \mathbb{C})$ where $T^{\mathbb{C}}$ is identified with the product of n copies of $\mathbb{C}^*$, each copy of $\mathbb{C}^*$ being endowed with the complex Poisson structure (2.1.6), calculating the brackets among the images $\widetilde{\beta}_1, \ldots, \widetilde{\beta}_\ell \in \mathbb{R}[T^{\mathbb{C}}]_{\mathbb{C}}^W$ of the generators $\beta_1, \ldots, \beta_\ell$ of the algebra $\mathbb{R}[V]_{\mathbb{C}}^W$ of W-invariants, we obtain a description

of the reduced Poisson algebra $(C^\infty(T^{\mathbb C}\big/W,\mathbb C),\{\,\cdot\,,\,\cdot\,\})$ in terms of generators of the algebra $C^\infty(T^{\mathbb C}\big/W,\mathbb C)\cong C^\infty(T^{\mathbb C},\mathbb C)^W$. Since $T^{\mathbb C}$ is actually a Kähler manifold, once a choice $f_1,\dots,f_k$ of complex invariants has been made, Poisson brackets of the kind $\{f_u,f_v\}$ and $\{\overline f_u,\overline f_v\}$ $(1\le u,v\le k)$ are necessarily zero.

3. Unitary and special unitary groups

We begin with the unitary group $K=\mathrm U(n)$.

(3.1) The complex structure of the adjoint quotient. The complexification $K^{\mathbb C}$ of $K=\mathrm U(n)$ is the full general linear group $\mathrm{GL}(n,\mathbb C)$, the standard maximal torus $T^{\mathbb C}\cong(\mathbb C^*)^n$ consists of the diagonal matrices in $\mathrm{GL}(n,\mathbb C)$, and the Weyl group W is the symmetric group S_n on n letters which acts on $(\mathbb C^*)^n$ by permutation of the factors. Since the action of the Weyl group on $T^{\mathbb C}\cong(\mathbb C^*)^n$ actually extends to the ambient complex vector space $\mathbb C^n$, the complex affine coordinate ring $\mathbb C[T^{\mathbb C}]$ may be written as

$$\mathbb C[T^{\mathbb C}]=\mathbb C[v_1,\dots,v_n,\sigma_n^{-1}]$$

where $\sigma_n=v_1\cdot\ldots\cdot v_n$ is the n'th elementary symmetric function, in such a way that the W-action permutes the $v_1,\dots,v_n$ and leaves invariant the coordinate function σ_n^{-1}. Thus we may carry out the construction in (2.2) above with $m=n+1$, $v_m=\sigma_n^{-1}$, and $V=\mathbb C^{n+1}$ the W-representation $V=\mathbb C^n\oplus\mathbb C$ where W acts on the copy of $\mathbb C^n$ by permutation of the factors and trivially on the residual copy $\mathbb C$. This yields $T^{\mathbb C}$ as the complex W-subvariety of $\mathbb C^{n+1}$ given by the single equation

$$v_1\cdot\ldots\cdot v_n\cdot v_{n+1}=1.$$

This observation, in turn, implies at once that the adjoint quotient may be realized by means of the map

$$(\sigma_1,\dots,\sigma_n)\colon(\mathbb C^*)^n\longrightarrow\mathbb C^{n-1}\times\mathbb C^* \tag{3.1.1}$$

whose target identifies the adjoint quotient $T^{\mathbb C}\big/S_n$ as a complex algebraic and hence complex analytic space (which may be realized as a complex affine variety in $\mathbb C^{n+1}$ as explained above). Here the constituents $\sigma_1,\dots,\sigma_n$ are the elementary symmetric functions; in fact, these are the restrictions to $T^{\mathbb C}$ of the characters of the fundamental (finite dimensional) representations of $\mathrm{GL}(n,\mathbb C)$.

After incorporation of the appropriate signs, which amounts to multiplying the j'th elementary symmetric function by $(-1)^j$, the description of the quotient map from $\mathrm{GL}(n,\mathbb C)$ to the adjoint quotient in terms of polynomials given in the introduction follows immediately from these observations. The stratum corresponding to the partition $n_1+\dots+n_k=n$ consists of the polynomials

$$(z-z_1)^{n_1}\dots(z-z_k)^{n_k}$$

in the variable z where $z_j\ne z_\ell$ when $j\ne\ell$ $(1\le j,\ell\le k)$; this stratum is a k-dimensional complex algebraic manifold. Indeed, the symmetric group S_k on k letters acts on the space of all

$$(z_1,\dots,z_1,z_2,\dots,z_2,\dots,z_k,\dots,z_k),\quad z_j\ne z_\ell, j\ne\ell\ (1\le j,\ell\le k),$$

in an obvious manner, and the projection to the S_n-quotient identifies the S_k-quotient with the corresponding stratum. In particular, the top stratum, that is, the open and dense stratum of complex dimension n, corresponds to the partition $n_1+\cdots+n_k=n$ of n where $k=n$ and where each $n_j=1$ $(1\leq j\leq n)$. The complement of the top stratum is, then, the *discriminant* variety. This is the variety over which the projection mapping (3.1.1) branches; it is given by the equation

$$D_n(1,-\sigma_1,\sigma_2,\dots,(-1)^n\sigma_n)=0$$

where $D_n(a_0,a_1,\dots,a_n)$ refers to the discriminant of the polynomial

$$P(w)=a_0w^n+a_1w^{n-1}+\cdots+a_n.$$

A detailed description of the closures of the strata as discriminant varieties may be found in [21].

(3.2) THE REAL COORDINATE RING OF THE ADJOINT QUOTIENT. We will spell out a description of the *complexification* $\mathbb{R}[T^{\mathbb{C}}/S_n]_{\mathbb{C}}$ of the real coordinate ring $\mathbb{R}[T^{\mathbb{C}}/S_n]$ of the adjoint quotient $T^{\mathbb{C}}/S_n$ of $K^{\mathbb{C}}=\mathrm{GL}(n,\mathbb{C})$, where $T^{\mathbb{C}}$ is the complexification of the maximal torus T in $K=\mathrm{U}(n)$.

Under the present circumstances, the complex algebra $\mathbb{R}[V]_{\mathbb{C}}$, cf. (2.2.3.1), is the polynomial algebra

$$\mathbb{R}[V]_{\mathbb{C}}=\mathbb{C}[z_1,\overline{z}_1,\dots,z_{n+1},\overline{z}_{n+1}]$$

in the variables $z_1,\overline{z}_1,\dots,z_{n+1},\overline{z}_{n+1}$, and the S_n-action on $\mathbb{R}[V]_{\mathbb{C}}$ permutes the $z_1,\dots,z_n$'s and the $\overline{z}_1,\dots,\overline{z}_n$'s separately and fixes z_{n+1} and $\overline{z}_{n+1}$. Hence the algebra of invariants $\mathbb{R}[V]_{\mathbb{C}}^{S_n}$ may be written as

$$\mathbb{R}[V]_{\mathbb{C}}^{S_n}\cong\mathbb{C}[z_1,\overline{z}_1,\dots,z_n,\overline{z}_n]^{S_n}\otimes\mathbb{C}[z_{n+1},\overline{z}_{n+1}].$$

Thus it suffices to determine the algebra $\mathbb{C}[z_1,\overline{z}_1,\dots,z_n,\overline{z}_n]^{S_n}$ of S_n-invariants which, in turn, is the algebra of invariants on the product of two copies of the standard permutation representation of the symmetric group S_n. We will refer to an algebra of this kind as an *algebra of bisymmetric functions*.

Systems of generators for an algebra of bisymmetric functions are well known and classical. To recall such a system, we will follow Ex. 5 in §5 of Chap. IV in [6]. Consider the polynomial ring $\mathbb{Q}[z_1,\dots,z_n,\overline{z}_1,\dots,\overline{z}_n]$ over the field $\mathbb{Q}$ of rational numbers, endowed with the obvious S_n-action which permutes the variables separately. For $r\geq 0$ and $s\geq 0$ such that $1\leq r+s\leq n$, let

$$\sigma_{(r,s)}(z_1,\dots,z_n,\overline{z}_1,\dots,\overline{z}_n) \tag{3.2.1}$$

be the S_n-orbit sum of the monomial $z_1\cdot\ldots\cdot z_r\overline{z}_1\cdot\ldots\cdot\overline{z}_s$; the function $\sigma_{(r,s)}$ is manifestly S_n-invariant, and we refer to a function of this kind as an *elementary bisymmetric function*. In degree m, $1\leq m\leq n$, the construction yields the $m+1$ bisymmetric functions $\sigma_{(m,0)}$, $\sigma_{(m-1,1)}$, $\dots$, $\sigma_{(0,m)}$ whence in degrees at most equal to n it yields altogether $\frac{n(n+3)}{2}$ elementary bisymmetric functions; in particular,

$$\begin{aligned}\sigma_{(m,0)}(z_1,\dots,z_n,\overline{z}_1,\dots,\overline{z}_n)&=\sigma_m(z_1,\dots,z_n),\\ \sigma_{(0,m)}(z_1,\dots,z_n,\overline{z}_1,\dots,\overline{z}_n)&=\sigma_m(\overline{z}_1,\dots,\overline{z}_n).\end{aligned}$$

According to a classical result, the ring $\mathbb{Q}[z_1,\ldots,z_n,\overline{z}_1,\ldots,\overline{z}_n]^{S_n}$ of bisymmetric functions (over the rationals) is generated by the elementary bisymmetric functions [6]. The result is actually more general and refers to general multisymmetric functions. Suitable multiples of the elementary bisymmetric functions arise from the elementary symmetric functions by *polarization* and, in terms of the polarizations of the elementary symmetric functions, the result, for general multisymmetric functions, is given in II.3 (p. 37) of [43]. Modern accounts may be found in [9], [11], [42].

The generators (3.2.1) will provide a satisfactory description of the real semialgebraic geometry of the adjoint quotient. On the other hand, we shall see in Theorem 3.4.1 below that the stratified symplectic Poisson algebra can much more easily be described in terms of the following system of multiplicative generators for the algebra of bisymmetric functions, which is entirely classical as well: For $r \geq 0$ and $s \geq 0$ such that $1 \leq r+s$, let

$$\tau_{(r,s)}(z_1,\ldots,z_n,\overline{z}_1,\ldots,\overline{z}_n) = \sum_{j=1}^{n} z_j^r \overline{z}_j^s; \tag{3.2.2}$$

plainly, $\tau_{(r,s)}$ is the S_n-orbit sum of the monomial $z_1^r\overline{z}_1^s$, such a function is manifestly S_n-invariant, and we refer to a function of this kind as a *bisymmetric power sum function*. Then $\tau_{(1,0)} = \sigma_{(1,0)} = \sigma_1$ and $\tau_{(0,1)} = \sigma_{(0,1)} = \overline{\sigma}_1$ and, in degree m, $2 \leq m \leq n$, the construction yields the $m+1$ bisymmetric functions $\tau_{(m,0)}$, $\tau_{(m-1,1)}$, $\ldots$, $\tau_{(0,m)}$ whence it yields altogether $\frac{n(n+3)}{2}$ bisymmetric power sum functions; in particular, when $\tau_m(w_1,\ldots,w_n) = w_1^m + \ldots + w_n^m$ denotes the ordinary m'th power sum function,

$$\begin{aligned}
\tau_{(m,0)}(z_1,\ldots,z_n,\overline{z}_1,\ldots,\overline{z}_n) &= \tau_m(z_1,\ldots,z_n) = z_1^m + \ldots + z_n^m,\\
\tau_{(0,m)}(z_1,\ldots,z_n,\overline{z}_1,\ldots,\overline{z}_n) &= \tau_m(\overline{z}_1,\ldots,\overline{z}_n) = \overline{z}_1^m + \ldots + \overline{z}_n^m,
\end{aligned}$$

and we will occasionally write τ_m instead of $\tau_{(m,0)}$ and $\overline{\tau}_m$ instead of $\tau_{(0,m)}$.

According to a variant of the already quoted classical result, over the rationals, the ring of bisymmetric functions in the variables z_1, $\ldots$, z_n, $\overline{z}_1$, $\ldots$, $\overline{z}_n$ is generated by the bisymmetric power sum functions $\tau_{(r,s)}$ with $1 \leq r+s \leq n$ as well [9], [11], [42].

Neither the generators (3.2.1) nor the generators (3.2.2) are free, that is, the algebra of bisymmetric functions is not a polynomial algebra when $n > 1$. It seems hard to find defining relations in the literature, in fact, we do not know of any reference for such relations except the description in Theorem 2 (1) of [42] where a procedure for constructing relations in terms of an appropropriate infinite system of generators is given. We will therefore explain how a system of defining relations can be derived from a classical result that has been known since the 19'th century, cf. [32].

Maintaining standard terminology, we will refer to the quotient field of the ring of bisymmetric functions (over the rationals) as the *field of bisymmetric functions* (over the rationals) . This field is plainly the field $\mathbb{Q}(z_1,\ldots,z_n,\overline{z}_1,\ldots,\overline{z}_n)^{S_n}$ of invariants in the field $\mathbb{Q}(z_1,\ldots,z_n,\overline{z}_1,\ldots,\overline{z}_n)$ of rational functions in z_1, $\ldots$,z_n, $\overline{z}_1$, $\ldots$,$\overline{z}_n$. The following result is classical, cf. [32].

Proposition 3.2.3. *The field $\mathbb{Q}(z_1,\ldots,z_n,\overline{z}_1,\ldots,\overline{z}_n)^{S_n}$ of bisymmetric functions is purely transcendantal of degree $2n$, having the functions*

$$\sigma_{(1,0)},\ \ldots,\sigma_{(n,0)},\ \sigma_{(0,1)},\ \sigma_{(1,1)},\ \ldots,\sigma_{(n-1,1)} \tag{3.2.4}$$

as free generators. In particular, each bisymmetric function can be written in a unique fashion as a rational function with rational coefficients in the functions (3.2.4)*. Likewise, the functions*

$$\tau_{(1,0)},\ \ldots,\tau_{(n,0)},\ \tau_{(0,1)},\ \tau_{(1,1)},\ \ldots,\tau_{(n-1,1)} \tag{3.2.5}$$

are free generators of the field of bisymmetric functions, whence each bisymmetric function can be written in a unique fashion as a rational function with rational coefficients in the functions (3.2.5)*.*

This is a special case of a more general classical result saying that the field of m-multi-S_n-symmetric functions is purely transcendental with mn generators; cf. [32]. This result, in turn, is a special case of the familiar fact that any symmetric power of a rational variety is again rational.

We sketch a proof, since we shall refer to the argument later in the paper.

Proof. We will only justify the claim involving the elementary bisymmetric functions.

In terms of the functions

$$\sigma_{(0,1)},\ \sigma_{(1,1)},\ \ldots,\sigma_{(n-1,1)}, z_1,\ldots,z_n, \tag{3.2.6}$$

for $1\leq j\leq n$, the generator $\overline{z}_j$ is given by the expression

$$\overline{z}_j = \frac{z_j^{n-1}\sigma_{(0,1)} - z_j^{n-2}\sigma_{(1,1)} + \ldots + (-1)^{n-1}\sigma_{(n-1,1)}}{(z_j-z_1)\cdot\ldots\cdot(z_j-z_n)} \tag{3.2.7}$$

where the denominator is a product of $n-1$ non-zero terms, that is, the j'th term which would formally be of the kind z_j-z_j does *not* occur. This yields the $\overline{z}_j$'s ($1\leq j\leq n$) as rational functions of the functions (3.2.6). Consequently the subfield of the field $\mathbb{Q}(z_1,\ldots,z_n,\overline{z}_1,\ldots,\overline{z}_n)$ generated by the functions (3.2.6) exhausts this field whence the field $\mathbb{Q}(z_1,\ldots,z_n,\overline{z}_1,\ldots,\overline{z}_n)^{S_n}$ of bisymmetric functions coincides with the field

$$\mathbb{Q}(\sigma_{(0,1)},\ \sigma_{(1,1)},\ \ldots,\sigma_{(n-1,1)}, z_1,\ldots,z_n)^{S_n}$$

which, in turn, comes down to the field

$$\mathbb{Q}(\sigma_{(0,1)},\ \sigma_{(1,1)},\ \ldots,\sigma_{(n-1,1)}, \sigma_1,\ldots,\sigma_n).$$

Since the field $\mathbb{Q}(z_1,\ldots,z_n,\overline{z}_1,\ldots,\overline{z}_n)$ has transcendence degree $2n$ and since it is a Galois extension of the field $\mathbb{Q}(\sigma_{(0,1)},\ \sigma_{(1,1)},\ \ldots,\sigma_{(n-1,1)}, \sigma_1,\ldots,\sigma_n)$ of bisymmetric functions, the latter has transcendence degree $2n$ as well, and it is in fact the field of rational functions in the generators $\sigma_{(0,1)},\ \ldots,\ \sigma_{(n-1,1)},\ \sigma_1,\ \ldots,\ \sigma_n$.

The assertion involving the bisymmetric power sum functions is established in a similar fashion. $\square$

Thus, for $2 \leq s \leq n$ and $0 \leq r \leq n$ with $r + s \leq n$, each generator $\sigma_{(r,s)}$ can be written in a unique fashion as a rational function

$$\sigma_{(r,s)} = \frac{\alpha_{(r,s)}}{\beta_{(r,s)}} \tag{3.2.8}$$

with rational coefficients, where the $\alpha_{(r,s)}$'s and $\beta_{(r,s)}$'s are polynomials in the functions (3.2.4) with rational coefficients such that each $\alpha_{(r,s)}$ is relatively prime to $\beta_{(r,s)}$. Indeed, each of the rational functions on the right-hand side of (3.2.8) may be determined in the following fashion: Given $2 \leq s \leq n$ and $0 \leq r \leq n$ with $r+s \leq n$, in the definition (3.2.1) of $\sigma_{(r,s)}$, for $1 \leq j \leq n$, substitute the right-hand side of (3.2.7) for each occurrence of $\overline{z}_j$. This yields each $\sigma_{(r,s)}$ as an S_n-invariant rational function of the functions (3.2.6), that is, when L denotes the field $\mathbb{Q}(\sigma_{(0,1)}, \sigma_{(1,1)}, \ldots, \sigma_{(n-1,1)})$ of rational functions in $\sigma_{(0,1)}$, $\sigma_{(1,1)}$, $\ldots$, $\sigma_{(n-1,1)}$, the procedure yields $\sigma_{(r,s)}$ as an S_n-invariant function in $L(z_1, \ldots, z_n)$, i. e. as a uniquely determined rational function in $L(\sigma_1, \ldots, \sigma_n)$. In practice, determining such a rational function explicitly may be a difficult endeavour, though; see (3.7) and (3.10) below.

Corollary 3.2.9. *For $s \geq 2$, the resulting $\frac{n(n-1)}{2}$ identities*

$$\beta_{(r,s)}\sigma_{(r,s)} = \alpha_{(r,s)}, \quad 0 \leq r \leq n, 2 \leq s \leq n, r + s \leq n, \tag{3.2.10}$$

are defining relations for the ring $\mathbb{Q}[z_1, \overline{z}_1, \ldots, z_n, \overline{z}_n]^{S_n}$ of bisymmetric functions (over the rationals).

Proof. Since $\mathbb{Q}[z_1, \overline{z}_1, \ldots, z_n, \overline{z}_n]^{S_n}$ is a domain, it injects into its quotient field. In the quotient field, the identities (3.2.10) are equivalent to the identities (3.2.8). □

At the risk of making a mountain out of a molehill we note that the complexification $\mathbb{R}[T^{\mathbb{C}}/S_n]_{\mathbb{C}}$ of the real coordinate ring $\mathbb{R}[T^{\mathbb{C}}/S_n]$ which we are looking for is simply the smallest subring of the field $\mathbb{Q}(z_1, \overline{z}_1, \ldots, z_n, \overline{z}_n)^{S_n}$ of bisymmetric functions containing $\mathbb{Q}[z_1, \overline{z}_1, \ldots, z_n, \overline{z}_n]^{S_n}$ and the inverses σ_n^{-1} and $\overline{\sigma}_n^{-1}$.

(3.3) THE REAL SEMIALGEBRAIC STRUCTURE OF THE ADJOINT QUOTIENT.
On $\mathbb{C}^n \times \mathbb{C}^n$, consider the complex conjugation given by $(\mathbf{z}_1, \mathbf{z}_2) \mapsto (\overline{\mathbf{z}}_2, \overline{\mathbf{z}}_1)$ $((\mathbf{z}_1, \mathbf{z}_2) \in \mathbb{C}^n \times \mathbb{C}^n)$. The real linear subspace invariant under this conjugation is the diagonal space

$$\Delta = \{(\mathbf{z}, \overline{\mathbf{z}}); \mathbf{z} \in \mathbb{C}^n\} \subseteq \mathbb{C}^n \times \mathbb{C}^n.$$

Relative to the obvious diagonal S_n-action on $\mathbb{C}^n \times \mathbb{C}^n$, Δ is an S_n-invariant subspace. In terms of the real coordinates x_j and y_j given by $z_j = x_j + iy_j$ $(1 \leq j \leq n)$, Δ is the real vector space with coordinates x_1, y_1, $\ldots$, x_n, y_n and, as a real non-singular variety, the complex torus $T^{\mathbb{C}}$ lies in Δ as the subspace of points $(x_1, y_1, \ldots, x_n, y_n)$ with $x_j + iy_j \neq 0$ for $1 \leq j \leq n$. The observations spelled out in Section 2 reduce the description of the structure of the adjoint quotient $T^{\mathbb{C}}/S_n$ to that of the orbit space Δ/S_n. To determine the latter we note first that, in view of Corollary 3.2.9, the complexification $\mathbb{R}[\Delta/S_n]_{\mathbb{C}}$ of the real coordinate ring $\mathbb{R}[\Delta/S_n]$ of the orbit space Δ/S_n is generated by the $d = \frac{n(n+3)}{2}$ invariants (3.2.1), subject to the relations (3.2.10).

To extract a description of the orbit space Δ/S_n as a real semialgebraic space, take the invariants (3.2.1) as coordinates on $\mathbb{C}^d$ and denote by $Y \subseteq \mathbb{C}^d$ the complex

affine variety given by the equations (3.2.10). For $j \neq k$ and $j+k \leq n$, $\sigma_{j,k}$ and $\sigma_{k,j}$ are here considered as independent complex variables. Let

$$p = (\sigma_{(1,0)}, \sigma_{(0,1)}, \sigma_{(2,0)}, \sigma_{(1,1)}, \dots, \sigma_{(0,n)}) \colon \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}^d \tag{3.3.1}$$

be the resulting HILBERT map, so that the complex variety $Y \subseteq \mathbb{C}^d$ coincides with the image of p.

The complex conjugation on $\mathbb{C}^n \times \mathbb{C}^n$ introduced above induces the complex conjugation on $\mathbb{C}^d$ which sends $\sigma_{(j,k)}$ to $\sigma_{(k,j)}$ $(1 \leq j+k \leq n)$ and $\sigma_{(\ell,\ell)}$ $(2 \leq 2\ell \leq n)$ to its complex conjugate $\overline{\sigma}_{(\ell,\ell)}$. Relative to this complex conjugation, embed $\mathbb{R}^d$ into $\mathbb{C}^d$ in the obvious manner, that is, as the subspace of invariants under this complex conjugation. Then the *real* categorical quotient $\widehat{\Delta/S_n}$ amounts to the intersection $Y \cap \mathbb{R}^d$, and the actual orbit space Δ/S_n lies in $\widehat{\Delta/S_n}$ as the semialgebraic set $p(\Delta)$ in $Y \cap \mathbb{R}^d$. We will now explain how inequalities defining this semialgebraic set may be derived; such inequalities are more easily spelled out in terms of the invariants (3.2.2).

For convenience we will consider the elements of the target $\mathbb{C}^d$ as column vectors. The invariants (3.3.2) furnish the Hilbert map

$$\Phi = \left[\tau_{(1,0)}, \tau_{(0,1)}, \tau_{(2,0)}, \tau_{(1,1)}, \tau_{(0,2)}, \dots, \tau_{(0,n)}\right]^{\mathrm{t}} \colon \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}^d \tag{3.3.2}$$

where $[\dots]^{\mathrm{t}}$ refers to the transpose of the row vector $[\dots]$. In the chosen coordinates, the Jacobian J_τ of Φ is the $(d \times (2n))$-matrix

$$J_\Phi = \left[\frac{\partial \Phi}{\partial z_1}, \frac{\partial \Phi}{\partial \overline{z}_1}, \dots, \frac{\partial \Phi}{\partial z_n}, \frac{\partial \Phi}{\partial \overline{z}_n}\right], \tag{3.3.3}$$

with column vectors $\frac{\partial \Phi}{\partial z_j}$ and $\frac{\partial \Phi}{\partial \overline{z}_j}$, for $1 \leq j \leq n$. A result in [33], see also Theorem 6.2 in [36], implies at once the following where B^{t} refers to the transpose of the matrix B.

Proposition 3.3.4. *Inequalities defining the quotient Δ/S_n as a real semi-algebraic space in $\mathbb{R}^d$ are given by the requirement that the hermitian $(d \times d)$-matrix $J_\Phi \overline{J}_\Phi^{\mathrm{t}}$ be positive semidefinite.* □

Notice that the condition that a given hermitian matrix B be positive semidefinite is equivalent to a simultaneous system of inequalities $\{B_\alpha \geq 0\}$ where B_α runs over the determinants of the hermitian minors of B.

(3.4) THE STRATIFIED SYMPLECTIC POISSON STRUCTURE OF THE ADJOINT QUOTIENT. For convenience, on $\mathrm{GL}(n, \mathbb{C})$, we will take the Kähler potential coming from the invariant quadratic form

$$\mathfrak{t} \to \mathbb{R}, \ (it_1, \dots, it_n) \longmapsto t_1^2 + \dots + t_n^2 \tag{3.4.1}$$

on the Lie algebra $\mathfrak{t}$ of the standard maximal torus of $\mathrm{U}(n)$.

Theorem 3.4.2. *The Poisson brackets among the multiplicative generators* $\tau_{(j,k)}$ $(0 \leq j \leq n, 0 \leq k \leq n, 1 \leq j+k \leq n)$ *of the ring* $\mathbb{Q}[z_1,\dots,z_n,\overline{z}_1,\dots,\overline{z}_n]^{S_n}$ *of bisymmetric functions are given by the formulas*

$$\frac{i}{2}\{\tau_{(j_1,k_1)},\tau_{(j_2,k_2)}\} = (j_1k_2 - j_2k_1)\tau_{(j_1+j_2,k_1+k_2)}. \tag{3.4.3}$$

In the statement of this theorem, when $j_1+j_2+k_1+k_2 > n$, the right-hand side of (3.4.3) is to be rewritten as a polynomial in terms of the multiplicative generators $\tau_{(j,k)}$, for $0 \leq j \leq n, 0 \leq k \leq n, 1 \leq j+k \leq n$. The theorem justifies the claim made earlier that the stratified symplectic Poisson structure is most conveniently described in terms of the power sum bisymmetric functions.

Proof. The assertion is established by a straightforward calculation relying on the formula (2.1.6). We leave the details to the reader. □

Theorem 3.4.2 yields the stratified symplectic Poisson algebra of the adjoint quotient for the unitary group $\mathrm{U}(n)$; for the group $\mathrm{SU}(n)$ a slight modification is necessary which we will explain shortly. The theorem has the following attractive consequence, the proof of which is immediate.

Corollary 3.4.4. *As a Poisson algebra, the ring* $\mathbb{Q}[z_1,\dots,z_n,\overline{z}_1,\dots,\overline{z}_n]^{S_n}$ *of bisymmetric functions is generated by the power sum functions* $\tau_1, \dots, \tau_n, \overline{\tau}_1, \dots, \overline{\tau}_n$. *That is to say: The ring of bisymmetric functions is generated by the power sum functions* $\tau_1, \dots, \tau_n, \overline{\tau}_1, \dots, \overline{\tau}_n$ *together with iterated Poisson brackets in these functions. Consequently the ring of bisymmetric functions is generated as well by the elementary symmetric functions* $\sigma_1, \dots, \sigma_n, \overline{\sigma}_1, \dots, \overline{\sigma}_n$ *together with iterated Poisson brackets in these functions.* □

This corollary establishes the theorem in the introduction for the special case where $K = \mathrm{U}(n)$.

Remark 3.4.5. There is an intimate relationship between the rank of the Poisson structure on the adjoint quotient and the real semi-algebraic structure. Indeed, let P be the $((2n)\times(2n))$-matrix

$$P = \begin{bmatrix} 0 & \{z_1,\overline{z}_1\} & 0 & 0 & \dots & 0 & 0 \\ \{\overline{z}_1,z_1\} & 0 & 0 & 0 & \dots & 0 & 0 \\ . & . & . & . & \dots & 0 & 0 \\ . & . & . & . & \dots & 0 & \{z_n,\overline{z}_n\} \\ . & . & . & . & \dots & \{\overline{z}_n,z_n\} & 0 \end{bmatrix}.$$

The skew-symmetric $(d\times d)$-matrix $\left[\{\tau_{(j_1,k_1)},\tau_{(j_2,k_2)}\}\right]$ of Poisson brackets among the multiplicative generators (3.2.2) may be written as

$$\left[\{\tau_{(j_1,k_1)},\tau_{(j_2,k_2)}\}\right] = J_\Phi P J_\Phi^{\mathrm{t}}. \tag{3.4.6}$$

The adjoint quotient $T^{\mathbb{C}}/S_n$ being realized in $\mathbb{C}^d$, given a point $(\mathbf{z},\overline{\mathbf{z}})$ of $T^{\mathbb{C}}$, at the point $\Phi(\mathbf{z},\overline{\mathbf{z}})$ of $T^{\mathbb{C}}/S_n \subseteq \mathbb{C}^d$, the rank of the Poisson structure is given by the rank of the matrix (3.4.6) at $\Phi(\mathbf{z},\overline{\mathbf{z}})$; this rank coincides with the rank of the matrix $J_\Phi(\mathbf{z},\overline{\mathbf{z}})$ since, at the point $(\mathbf{z},\overline{\mathbf{z}})$ of $T^{\mathbb{C}}$, the matrix P has maximal rank. Consequently the

rank of the Poisson structure at the point $\Phi(\mathbf{z},\overline{\mathbf{z}})$ of $T^{\mathbb{C}}\big/S_n \subseteq \mathbb{C}^d$ coincides with the rank of the $(d\times d)$- matrix $J_\Phi \overline{J}^{\mathrm{t}}_\Phi$ coming into play in (3.3) above. This implies that certain *unions of closures of strata may be described by requiring that some of the inequalities giving the real semi-algebraic structure be equalities.* Given a stratum Y of $T^{\mathbb{C}}\big/S_n$, when this stratum has dimension $2k$, as a smooth symplectic manifold the stratum arises as the base of a regular finite covering having as total space a smooth $(2k)$-dimensional symplectic submanifold of $T^{\mathbb{C}}$ whence the Poisson bracket, restricted to the stratum Y, necessarily has rank $2k$.

(3.5) The Kähler potential (1.2), restricted to $T^{\mathbb{C}}$, can be written as a real analytic function of the invariants (3.2.1) or (3.2.2) and hence descends to a real analytic function κ_{red} on the adjoint quotient, that is, determines an element of $C^\omega(T^{\mathbb{C}}\big/S_n)$. We shall illustrate this for $K = \mathrm{SU}(2)$ below.

(3.6) $K = \mathrm{U}(2)$; now the map (3.1.1) (for $n = 2$) induces a complex algebraic isomorphism from the adjoint quotient onto the complex manifold $\mathbb{C}\times\mathbb{C}^*$, with complex coordinates σ_1 and σ_2, $\sigma_2 \neq 0$. There is one lower stratum which corresponds to the center of $\mathrm{U}(2)$ and arises from the diagonal in $T^{\mathbb{C}} \cong \mathbb{C}^*\times\mathbb{C}^*$. As an affine complex variety, this stratum is the *discriminant variety*, and an equation for it arises from rewriting the S_2-invariant equation $(z_1 - z_2)^2 = 0$ in terms of the S_2-invariants σ_1 and σ_2; this yields the familiar equation

$$\sigma_1^2 - 4\sigma_2 = 0 \ \ (\sigma_2 \neq 0), \tag{3.6.1}$$

the left-hand side of this equation being the discriminant of the polynomial $P(z) = z^2 - \sigma_1 z + \sigma_2$. Thus geometrically the lower stratum amounts to a (complex) parabola in the adjoint quotient $\mathbb{C}\times\mathbb{C}^*$, that is, to a parabola with the origin removed.

To describe the Poisson structure on the complexification $\mathbb{R}[T^{\mathbb{C}}/S_2]_{\mathbb{C}}$ of the real coordinate ring $\mathbb{R}[T^{\mathbb{C}}/S_2]$ of the adjoint quotient $T^{\mathbb{C}}/S_2$, we note first that, in view of the observations spelled out in Subsection 3.2 above, the algebra of S_2-invariants $\mathbb{C}[z_1,\overline{z}_1,z_2,\overline{z}_2]^{S_2}$ is generated by the elementary bisymmetric functions

$$\sigma_1 = z_1 + z_2,\ \ \overline{\sigma}_1 = \overline{z}_1 + \overline{z}_2,\ \ \sigma_2 = z_1 z_2,\ \ \overline{\sigma}_2 = \overline{z}_1\overline{z}_2,\ \ \sigma = \sigma_{(1,1)} = z_1\overline{z}_2 + z_2\overline{z}_1; \tag{3.6.2}$$

moreover, these generators are subject to the single defining relation

$$(\sigma_1^2 - 4\sigma_2)(\overline{\sigma}_1^2 - 4\overline{\sigma}_2) = (\sigma_1\overline{\sigma}_1 - 2\sigma)^2. \tag{3.6.3}$$

Indeed, in view of the formula (3.2.7),

$$\overline{\sigma}_2 = \overline{z}_1\overline{z}_2 = -\frac{(z_1\overline{\sigma}_1 - \sigma)(z_2\overline{\sigma}_1 - \sigma)}{(z_1 - z_2)^2}. \tag{3.6.4}$$

A calculation shows that this identity is equivalent to (3.6.3). In terms of the five bisymmetric power sum functions $\tau_1 = \tau_{1,0}$, $\overline{\tau}_1 = \tau_{0,1}$, $\tau = \tau_{1,1}$, $\tau_2 = \tau_{2,0}$, $\overline{\tau}_2 = \tau_{0,2}$, the (algebraic) Poisson structure is given by Theorem 3.4.2, and a straightforward calculation yields expressions for the Poisson structure in terms of the generators (3.6.2).

Under the present circumstances, other calculations yield

$$J_\Phi \overline{J}^{\mathrm{t}}_\Phi = \begin{bmatrix} 2 & \overline{\tau}_1 & 0 & 0 & \tau_1 \\ \tau_1 & \tau & 0 & 0 & \tau_2 \\ 0 & 0 & 2 & \tau_1 & \overline{\tau}_1 \\ 0 & 0 & \overline{\tau}_1 & \tau & \overline{\tau}_2 \\ \overline{\tau}_1 & \overline{\tau}_2 & \tau_1 & \tau_2 & 2\tau \end{bmatrix} \tag{3.6.5}$$

and

$$\left| J_\Phi \overline{J}^{\mathrm{t}}_\Phi \right| = 16(2\tau - \tau_1\overline{\tau}_1)((2\tau - \tau_1\overline{\tau}_1)^2 - (2\tau_2 - \tau_1^2)(2\overline{\tau}_2 - \overline{\tau}_1^2)). \tag{3.6.6}$$

Furthermore, straightforward calculation yields

$$\begin{aligned} 2\tau - \tau_1\overline{\tau}_1 &= \sigma_1\overline{\sigma}_1 - 2\sigma = (z_1 - z_2)(\overline{z}_1 - \overline{z}_2) \\ 2\tau_2 - \tau_1^2 &= \sigma_1^2 - 4\sigma_2 \\ 2\overline{\tau}_2 - \overline{\tau}_1^2 &= \overline{\sigma}_1^2 - 4\overline{\sigma}_2 \\ (2\tau - \tau_1\overline{\tau}_1)^2 - (2\tau_2 - \tau_1^2)(2\overline{\tau}_2 - \overline{\tau}_1^2) &= (\sigma_1\overline{\sigma}_1 - 2\sigma)^2 - (\sigma_1^2 - 4\sigma_2)(\overline{\sigma}_1^2 - 4\overline{\sigma}_2) \end{aligned}$$

whence

$$\left| J_\Phi \overline{J}^{\mathrm{t}}_\Phi \right| = 16(\sigma_1\overline{\sigma}_1 - 2\sigma)((\sigma_1\overline{\sigma}_1 - 2\sigma)^2 - (\sigma_1^2 - 4\sigma_2)(\overline{\sigma}_1^2 - 4\overline{\sigma}_2)). \tag{3.6.7}$$

In view of the relation (3.6.3), the second factor on the right-hand side of this equation is identically zero. However, the lower stratum is characterized by the equation

$$\sigma_1\overline{\sigma}_1 - 2\sigma = 0 \tag{3.6.8}$$

whence we see that, on the lower stratum, the Poisson structure necessarily has rank 2. By Proposition 3.3.4, inequalities characterizing the real semialgebraic structure are given by requiring that, for $\ell = 1, 2, 3$, the symmetric $(\ell \times \ell)$-minors of (3.6.5) be non-negative. This leads to (not necessarily independent) inequalities of the kind

$$\tau \geq 0 \tag{3.6.9.1}$$

$$\begin{vmatrix} 2 & \overline{\tau}_1 \\ \tau_1 & \tau \end{vmatrix} = 2\tau - \tau_1\overline{\tau}_1 = \sigma_1\overline{\sigma}_1 - 2\sigma \geq 0 \tag{3.6.9.2}$$

$$\begin{vmatrix} 2 & \tau_1 & \overline{\tau}_1 \\ \overline{\tau}_1 & \tau & \overline{\tau}_2 \\ \tau_1 & \tau_2 & 2\tau \end{vmatrix} = 4\tau - 2\tau_2\overline{\tau}_2 - 3\tau\tau_1\overline{\tau}_1 + \tau_1^2\overline{\tau}_2 + \overline{\tau}_1^2\tau_2 \geq 0 \tag{3.6.9.3}$$

$$\begin{vmatrix} \tau & 0 & 0 & \tau_2 \\ 0 & 2 & \tau_1 & \overline{\tau}_1 \\ 0 & \overline{\tau}_1 & \tau & \overline{\tau}_2 \\ \overline{\tau}_2 & \tau_1 & \tau_2 & 2\tau \end{vmatrix} = (\tau^2 - \tau_2\overline{\tau}_2)(2\tau - \tau_1\overline{\tau}_1) \geq 0 \tag{3.6.9.4}$$

etc. We note that, in view of the defining relation (3.6.3),

$$\begin{aligned} \tau^2 - \tau_2\overline{\tau}_2 &= \frac{1}{4}((\sigma_1\overline{\sigma}_1 - 2\sigma)^2 - (\sigma_1^2 - 4\sigma_2)(\overline{\sigma}_1^2 - 4\overline{\sigma}_2)) + \sigma_2\overline{\sigma}_1^2 + \overline{\sigma}_2\sigma_1^2 - \sigma_1\overline{\sigma}_1\sigma \\ &= \sigma_2\overline{\sigma}_1^2 + \overline{\sigma}_2\sigma_1^2 - \sigma_1\overline{\sigma}_1\sigma = 4\sigma_2\overline{\sigma}_2 - \sigma^2 \end{aligned}$$

whence the inequality (3.6.9.4) is equivalent to

$$(4\sigma_2\overline{\sigma}_2 - \sigma^2)(\sigma_1\overline{\sigma}_1 - 2\sigma) \geq 0. \tag{3.6.9.5}$$

To elucidate the real structure of the quotient, in particular its semialgebraicity, entirely in terms of appropriate real data, let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, and write $\sigma_1 = X + iY$ and $\sigma_2 = U + iV$, so that

$$\begin{gathered} X = x_1 + x_2,\ \ Y = y_1 + y_2,\ \ U = x_1x_2 - y_1y_2,\ \ V = x_1y_2 + x_2y_1, \\ \sigma = 2(x_1x_2 + y_1y_2). \end{gathered} \tag{3.6.10}$$

In view of the observations spelled out in Section 3 above, the algebra of S_2-invariants $\mathbb{Q}[x_1, y_1, x_2, y_2]^{S_2}$ is generated by X, Y, U, V, and σ, and a straightforward calculation shows that the defining relation (3.6.3), written out in terms of these generators, has the form

$$(X^2 - Y^2 - 4U)^2 + 4(XY - 2V)^2 = (X^2 + Y^2 - 2\sigma)^2. \tag{3.6.11}$$

Thus the *real categorical* quotient of $T^{\mathbb{C}} \cong \mathbb{C}^* \times \mathbb{C}^*$ modulo $W = S_2$ is realized in $\mathbb{R}^5$ with coordinates X, Y, U, V, σ as the real variety given by the equation (3.6.11) with $U \neq 0$ and, since $\sigma_1\overline{\sigma}_1 = X^2 + Y^2$, in view of the inequalities (3.6.9.2) and (3.6.9.5), the quotient $T^{\mathbb{C}}\big/W$ we are really interested in is the semialgebraic subspace of the real categorical quotient given by the inequalities

$$X^2 + Y^2 - 2\sigma \geq 0 \tag{3.6.14}$$

$$4(U^2 + V^2) - \sigma^2 \geq 0. \tag{3.6.15}$$

It is straightforward to rewrite the Poisson brackets (3.4.3) in terms of the generators (3.6.2) and the generators (3.6.10). For example

$$\begin{aligned} \frac{i}{2}\{\sigma_1, \overline{\sigma}_1\} &= \{X, Y\} = \tau = X^2 + Y^2 - \sigma, \\ \frac{i}{2}\{\sigma_2, \overline{\sigma}_2\} &= \{U, V\} = 2\sigma_2\overline{\sigma}_2 = 2(U^2 + V^2), \\ \frac{i}{2}\{\sigma_1, \overline{\sigma}_2\} &= \sigma_1\overline{\sigma}_2, \end{aligned}$$

etc. We leave the details to the reader.

(3.7) $K = \mathrm{U}(3)$; in this case the map (3.1.1) (for $n = 3$) induces a complex algebraic isomorphism from the adjoint quotient $T^{\mathbb{C}}\big/S_3 \cong (\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*)\big/S_3$ onto the complex algebraic manifold $\mathbb{C} \times \mathbb{C} \times \mathbb{C}^*$, with complex coordinates σ_1, σ_2, and σ_3, $\sigma_3 \neq 0$. The adjoint quotient has three strata; in terms of the maximal torus $T^{\mathbb{C}} \cong \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$, these arise from matrices $\mathrm{diag}(z_1, z_2, z_3) \in T^{\mathbb{C}}$ of the kind $z_1 \neq z_2 \neq z_3 \neq z_1$, $z_1 = z_2 \neq z_3$, and $z_1 = z_2 = z_3$. The complement of the 3-dimensional stratum is the corresponding *discriminant variety*, that is, the affine variety given by the equation

$$D_3(1, -\sigma_1, \sigma_2, -\sigma_3) = 0$$

where

$$D_3(a_0,a_1,a_2,a_3) = a_1^2a_2^2 - 4a_0a_2^3 - 4a_1^3a_3 - 27a_0^2a_3^2 + 18a_0a_1a_2a_3 \tag{3.7.1}$$

refers to the discriminant of the polynomial $a_0w^3+a_1w^2+a_2w+a_3$. The discriminant variety is the closure of the 2-dimensional stratum. Likewise, when

$$D_2(b_0,b_1,b_2) = b_1^2 - 4b_0b_2 \tag{3.7.2}$$

refers to the discriminant of the polynomial $b_0w^2+b_1w+b_2$, the 1-dimensional stratum is the affine curve given by the additional equation

$$D_2(3,-2\sigma_1,\sigma_2) = 0.$$

This is just the cubic complex curve given by the parametrization $v \mapsto (3v, 3v^2, v^3)$ $(v \in \mathbb{C})$.

In view of the observations spelled out in (3.2) above, the complexification $\mathbb{R}[T^{\mathbb{C}}/S_3]_{\mathbb{C}}$ of the real coordinate ring $\mathbb{R}[T^{\mathbb{C}}/S_3]$ of the adjoint quotient $T^{\mathbb{C}}/S_3$ is the algebra of S_3-invariants $\mathbb{C}[z_1,\overline{z}_1,z_2,\overline{z}_2,z_3,\overline{z}_3]^{S_3}$, and this algebra is generated by the elementary bisymmetric functions

$$\begin{gathered}
\sigma_1 = \sigma_{(1,0)},\ \ \overline{\sigma}_1 = \sigma_{(0,1)},\ \ \sigma_2 = \sigma_{(2,0)},\ \ \overline{\sigma}_2 = \sigma_{(0,2)},\ \ \sigma_3 = \sigma_{(3,0)},\ \ \overline{\sigma}_3 = \sigma_{(0,3)},\\
\sigma = \sigma_{(1,1)} = z_1\overline{z}_2 + z_2\overline{z}_1 + z_2\overline{z}_3 + z_3\overline{z}_2 + z_3\overline{z}_1 + z_1\overline{z}_3,\\
\rho = \sigma_{(2,1)} = (z_1\overline{z}_2z_3 + z_2\overline{z}_3z_1 + z_3\overline{z}_1z_2),\\
\overline{\rho} = \sigma_{(1,2)} = (z_1\overline{z}_2\overline{z}_3 + z_2\overline{z}_3\overline{z}_1 + z_3\overline{z}_1\overline{z}_2).
\end{gathered} \tag{3.7.3}$$

Moreover, in view of Proposition 3.2.3, the field of bisymmetric functions is freely generated by σ_1, $\overline{\sigma}_1$, σ_2, σ_3, σ and ρ.

In the present special case where $n=3$, the formula (3.2.7) comes down to

$$\overline{z}_1 = \frac{z_1^2\overline{\sigma}_1 - z_1\sigma + \rho}{(z_1-z_2)(z_1-z_3)},\ \ \overline{z}_2 = \frac{z_2^2\overline{\sigma}_1 - z_2\sigma + \rho}{(z_2-z_1)(z_2-z_3)},\ \ \overline{z}_3 = \frac{z_3^2\overline{\sigma}_1 - z_3\sigma + \rho}{(z_3-z_1)(z_3-z_2)}.$$

Thus the symmetric function $\overline{\sigma}_2 = \overline{z}_1\overline{z}_2 + \overline{z}_1\overline{z}_3 + \overline{z}_2\overline{z}_3$ takes the form

$$\overline{\sigma}_2 = \frac{\sum_{j \bmod 3}(z_{j+2}-z_j)(z_{j+2}-z_{j+1})(z_j^2\overline{\sigma}_1 - z_j\sigma+\rho)(z_{j+1}^2\overline{\sigma}_1 - z_{j+1}\sigma+\rho)}{D_3(1,-\sigma_1,\sigma_2,-\sigma_3)}, \tag{3.7.4}$$

and this identity yields a relation of the kind

$$D_3(1,-\sigma_1,\sigma_2,-\sigma_3)\overline{\sigma}_2 = \alpha(\sigma_1,\sigma_2,\sigma_3,\overline{\sigma}_1,\sigma,\rho) \tag{3.7.5}$$

where $\alpha(\sigma_1,\sigma_2,\sigma_3,\overline{\sigma}_1,\sigma,\rho)$ is a polynomial in the indicated generators. A tedious but straightforward calculation involving the Newton polynomials yields the following expression for α:

$$\begin{aligned}
\alpha(\sigma_1,\sigma_2,\sigma_3,\overline{\sigma}_1,\sigma,\rho) &= (9\sigma_3^2 + \sigma_2^3 - 4\sigma_1\sigma_2\sigma_3)\overline{\sigma}_1^2\\
&\quad + (4\sigma_1^2\sigma_3 - 3\sigma_2\sigma_3 - \sigma_1\sigma_2^2)\overline{\sigma}_1\sigma\\
&\quad + (6\sigma_1\sigma_3 - \sigma_2^2)\overline{\sigma}_1\rho + (\sigma_2^2 - 3\sigma_1\sigma_3)\sigma^2\\
&\quad + (9\sigma_3 - \sigma_1\sigma_2)\sigma\rho + (\sigma_1^2 - 3\sigma_2)\rho^2
\end{aligned} \tag{3.7.6}$$

Likewise, the symmetric function $\overline{\sigma}_3 = \overline{z}_1\overline{z}_2\overline{z}_3$ takes the form

$$\overline{\sigma}_3 = \frac{(z_1^2\overline{\sigma}_1 - z_1\sigma + \rho)(z_2^2\overline{\sigma}_1 - z_2\sigma + \rho)(z_3^2\overline{\sigma}_1 - z_3\sigma + \rho)}{D_3(1, -\sigma_1, \sigma_2, -\sigma_3)}, \tag{3.7.7}$$

and this identity yields a relation of the kind

$$D_3(1, -\sigma_1, \sigma_2, -\sigma_3)\overline{\sigma}_3 = \beta(\sigma_1, \sigma_2, \sigma_3, \overline{\sigma}_1, \sigma, \rho) \tag{3.7.8}$$

where $\beta(\sigma_1, \sigma_2, \sigma_3, \overline{\sigma}_1, \sigma, \rho)$ is a polynomial in the indicated generators. Again a tedious but straightforward calculation yields the following expression for β:

$$\begin{aligned}\beta(\sigma_1, \sigma_2, \sigma_3, \overline{\sigma}_1, \sigma, \rho) = {}& \sigma_3^2\overline{\sigma}_1^3 - \sigma_2\sigma_3\overline{\sigma}_1^2\sigma + (\sigma_2^2 - 2\sigma_1\sigma_3)\overline{\sigma}_1^2\rho + \sigma_1\sigma_3\overline{\sigma}_1\sigma^2 \\ & - ((\sigma_1^2 - 2\sigma_2)\sigma_1 - \sigma_1^3 + 3\sigma_1\sigma_2 - 3\sigma_3)\overline{\sigma}_1\sigma\rho \\ & - \sigma_3\sigma^3 + (\sigma_1^2 - 2\sigma_2)\overline{\sigma}_1\rho^2 \\ & + \sigma_2\sigma^2\rho - \sigma_1\sigma\rho^2 + \rho^3\end{aligned} \tag{3.7.9}$$

Moreover, a calculation yields the relation

$$(\sigma_1^2 - 4\sigma_2)(\overline{\sigma}_1^2 - 4\overline{\sigma}_2) = (\sigma_1\overline{\sigma}_1 - 2\sigma)^2 + 2\rho\overline{\sigma}_1 + 2\overline{\rho}\sigma_1. \tag{3.7.10}$$

Since substituting the right-hand side of (3.7.4) for $\overline{\sigma}_2$ in (3.7.10) yields an identity which, in turn, yields $\overline{\rho} = \sigma_{(1,2)}$ as a rational function of the generators σ_1, σ_2, σ_3, $\overline{\sigma}_1$, σ, ρ, we conclude that, by virtue of Corollary 3.2.9, the relations (3.7.5), (3.7.8), and (3.7.10) are *defining* relations for the algebra $\mathbb{C}[z_1, \overline{z}_1, z_2, \overline{z}_2, z_3, \overline{z}_3]^{S_3}$ of S_3-invariants. We note that when we set the variables z_3 and $\overline{z}_3$ equal to zero, that is, under the obvious surjection

$$\mathbb{C}[z_1, \overline{z}_1, z_2, \overline{z}_2, z_3, \overline{z}_3]^{S_3} \longrightarrow \mathbb{C}[z_1, \overline{z}_1, z_2, \overline{z}_2]^{S_2},$$

the relation (3.7.10) passes to the relation (3.6.3).

Straightforward calculation yields the Poisson structure given by Theorem 3.4.2 in terms of the generators (3.7.3) and, furthermore, the corresponding real semialgebraic structure made explicit in (3.3) above in terms of appropriate equations and inequalities, cf. Proposition 3.3.4. We refrain from spelling out details.

(3.8) $K = \mathrm{SU}(n)$. The complexification $\mathrm{SU}(n)^{\mathbb{C}}$ equals the group $\mathrm{SL}(n, \mathbb{C})$, the standard maximal torus $T^{\mathbb{C}} \cong (\mathbb{C}^*)^{n-1}$ consists of the diagonal matrices in $\mathrm{SL}(n, \mathbb{C})$, that is, diagonal matrices of the kind $\mathrm{diag}(w_1, \ldots, w_n)$ with $w_1 \cdot \ldots \cdot w_n = 1$, and the Weyl group W is the symmetric group S_n on n letters which acts on $T^{\mathbb{C}}$ by permutation of the diagonal entries $w_1, \ldots, w_n$ of the matrices $\mathrm{diag}(w_1, \ldots, w_n)$. Hence the complex affine coordinate ring $\mathbb{C}[T^{\mathbb{C}}]$ may be written as

$$\mathbb{C}[T^{\mathbb{C}}] \cong \mathbb{C}[v_1, \ldots, v_n]/(\sigma_n - 1)$$

in such a way that the W-action permutes the $v_1, \ldots, v_n$. Thus we can carry out the construction in (2.2) above with $m = n$ and $V = \mathbb{C}^n$ the S_n-representation where S_n acts on $\mathbb{C}^n$ by permutation of the factors $\mathbb{C}$ so that complex algebraically the quotient

V/S_n amounts to the n'th symmetric power of a copy of $\mathbb{C}$. This construction yields $T^{\mathbb{C}}$ as the complex S_n-subvariety of $\mathbb{C}^n$ given by the single equation

$$v_1 \cdot \ldots \cdot v_n = 1.$$

This observation, in turn, implies at once that the adjoint quotient $T^{\mathbb{C}}/S_n$ may be realized by means of the map

$$(\sigma_1, \ldots, \sigma_{n-1}) \colon T^{\mathbb{C}} \longrightarrow \mathbb{C}^{n-1}. \tag{3.8.1}$$

This description of the categorical quotient of $\mathrm{SL}(n, \mathbb{C})$ as an affine $(n-1)$-dimensional space is consistent with Steinberg's observation quoted earlier; indeed, the elementary symmetric functions $\sigma_1, \ldots, \sigma_{n-1}$ are the restrictions to $T^{\mathbb{C}}$ of the characters of the fundamental (finite dimensional) representations of $\mathrm{SL}(n, \mathbb{C})$.

In view of the observations spelled out in Subsection 3.2 above, the complexification $\mathbb{R}[T^{\mathbb{C}}/S_n]_{\mathbb{C}}$ of the real coordinate ring $\mathbb{R}[T^{\mathbb{C}}/S_n]$ of the adjoint quotient under discussion is generated by the $\frac{n(n+3)}{2}$ invariants of the kind (3.2.1); however, in view of the relations $\sigma_n = 1$ and $\overline{\sigma}_n = 1$, we may discard the generators σ_n and $\overline{\sigma}_n$. Inequalities characterizing the real semialgebraic structure may be derived from the corresponding inequalities for the case where $K = \mathrm{U}(n)$.

We will now explain how a modification of the statement of Theorem 3.4.2 yields the stratified symplectic Poisson algebra on the adjoint quotient $T^{\mathbb{C}}/S_n$. It is obvious that the generators (3.2.2) yield multiplicative generators for the complexification of the real coordinate ring of this quotient as well. The problem is that we cannot simply restrict the Poisson brackets to $\mathrm{SL}(n, \mathbb{C})$, viewed as a subgroup of $\mathrm{GL}(n, \mathbb{C})$ since, in the language of *constrained systems*, $\mathrm{SL}(n, \mathbb{C})$ is a *second class* constraint in $\mathrm{GL}(n, \mathbb{C})$, cf. [10]. On $\mathrm{SL}(n, \mathbb{C})$ we take the Kähler potential given by restriction of the Kähler potential on $\mathrm{GL}(n, \mathbb{C})$ determined by (3.4.1) above. Then the embedding of $\mathrm{SL}(n, \mathbb{C})$ into $\mathrm{GL}(n, \mathbb{C})$ is one of Kähler manifolds. In view of the discussion in Section 2 above, the complexification $\mathbb{R}[T^{\mathbb{C}}/S_n]_{\mathbb{C}}$ of the real coordinate ring of the adjoint quotient $T^{\mathbb{C}}/S_n$ amounts to the ring of invariants

$$\big(\mathbb{C}[z_1, \ldots, z_n, \overline{z}_1, \ldots, \overline{z}_n]/(\sigma_n - 1, \overline{\sigma}_n - 1)\big)^{S_n}.$$

We will write the Poisson bracket on the ring $(\mathbb{C}[z_1, \ldots, z_n, \overline{z}_1, \ldots, \overline{z}_n])^{S_n}$ given in Theorem 3.4.2 above as $\{\{\,\cdot\,,\,\cdot\,\}\}$.

Theorem 3.8.2. *On the complexification $\mathbb{R}[T^{\mathbb{C}}/S_n]_{\mathbb{C}}$ of the real coordinate ring of the adjoint quotient $T^{\mathbb{C}}/S_n$, the Poisson brackets among the multiplicative generators*

$$\tau_{(j,k)}, \quad 0 \leq j \leq n, 0 \leq k \leq n, 1 \leq j+k \leq n,$$

are given by the formulas

$$\begin{aligned}\frac{i}{2}\{\tau_{(j_1,k_1)}, \tau_{(j_2,k_2)}\} = {} & \frac{i}{2}\{\{\tau_{(j_1,k_1)}, \tau_{(j_2,k_2)}\}\} \\ & - \frac{i}{2}\frac{\{\{\tau_{(j_1,k_1)}, \overline{\sigma}_n\}\}\{\{\sigma_n, \tau_{(j_2,k_2)}\}\}}{\{\{\sigma_n, \overline{\sigma}_n\}\}} \\ & - \frac{i}{2}\frac{\{\{\tau_{(j_1,k_1)}, \sigma_n\}\}\{\{\overline{\sigma}_n, \tau_{(j_2,k_2)}\}\}}{\{\{\overline{\sigma}_n, \sigma_n\}\}}.\end{aligned} \tag{3.8.3}$$

Proof. We recall that, in general, given a smooth symplectic manifold and a smooth symplectic submanifold, the Poisson bracket on the submanifold is induced from the *Dirac bracket* on the ambient manifold, cf. [10]. Hence, by virtue of Theorem 3.4.2, the statement of the Theorem is a consequence of the fact that, given two functions f and h on the complexification $\widetilde{T}^{\mathbb{C}}$ $(\subseteq \mathrm{GL}(n,\mathbb{C}))$ of the standard maximal torus $\widetilde{T}$ in $\mathrm{U}(n)$, relative to the constraints $\sigma_n = 1$ and $\overline{\sigma}_n = 1$, the *Dirac bracket* $\{f,h\}$ is given by

$$\{f,h\} = \{\{f,h\}\} - \frac{\{\{f,\sigma_n-1\}\}\{\{\overline{\sigma}_n-1,h\}\}}{\{\{\overline{\sigma}_n-1,\sigma_n-1\}\}} - \frac{\{\{f,\overline{\sigma}_n-1\}\}\{\{\sigma_n-1,h\}\}}{\{\{\sigma_n-1,\overline{\sigma}_n-1\}\}}. \quad \square$$

The following is immediate, cf. Corollary 3.4.4.

Corollary 3.8.4. *As a Poisson algebra, the complexification $\mathbb{R}[T^{\mathbb{C}}/S_n]_{\mathbb{C}}$ of the real coordinate ring of the adjoint quotient $T^{\mathbb{C}}/S_n$ for* $\mathrm{SU}(n)$ *is generated by the power sum functions $\tau_1, \ldots, \tau_n, \overline{\tau}_1, \ldots, \overline{\tau}_n$ and hence by the elementary symmetric functions $\sigma_1, \ldots, \sigma_{n-1}, \overline{\sigma}_1, \ldots, \overline{\sigma}_{n-1}$.* $\square$

Since the elementary symmetric functions are the characters of the fundamental representations of the group $\mathrm{SU}(n)$, the statement of the theorem in the introduction for $K = \mathrm{SU}(n)$ is an immediate consequence of this corollary.

(3.9) $K = \mathrm{SU}(2)$: We will now make the stratified Kähler structure explicit for $n = 2$, and we will spell out the reduced Kähler potential. The standard maximal torus $T^{\mathbb{C}}$ in $K^{\mathbb{C}} = \mathrm{SL}(2,\mathbb{C})$ is the group of diagonal matrices of the kind $\begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix}$ $(z \in \mathbb{C}, z \neq 0)$ in $\mathrm{SL}(2,\mathbb{C})$. Under the identification of $\mathbb{C}^*$ with $T^{\mathbb{C}}$ given by the assignment to $z \in \mathbb{C}^*$ of the diagonal matrix $\mathrm{diag}(z, z^{-1})$, the action of the Weyl group $W = S_2$ on $T^{\mathbb{C}}$ amounts to the S_2-action on $\mathbb{C}^*$ where the non-trivial involution acts by inversion. The first elementary symmetric function σ_1, that is, the character of the defining representation, comes down to the holomorphic function

$$\sigma_1 = \chi \colon \mathbb{C}^* \longrightarrow \mathbb{C}, \quad \chi(z) = z + z^{-1}, \quad z \in \mathbb{C},$$

which identifies the quotient $T^{\mathbb{C}}\big/S_2 \cong \mathbb{C}^*\big/S_2$ with the affine complex line $\mathbb{C}$.

Next we elucidate the real structure. According to the general construction spelled out in (3.8), the complexification $\mathbb{R}[T^{\mathbb{C}}/S_n]_{\mathbb{C}}$ of the real coordinate ring $\mathbb{R}[T^{\mathbb{C}}/S_n]$ is generated by $Z = \sigma_1 = z + z^{-1}$, $\overline{Z} = \overline{\sigma}_1 = \overline{z} + \overline{z}^{-1}$, and $\sigma = \frac{z}{\overline{z}} + \frac{\overline{z}}{z}$. In view of (3.6.3), these generators are subject to the relation

$$\sigma^2 + Z^2 + \overline{Z}^2 - Z\overline{Z}\sigma - 4 = 0. \tag{3.9.1}$$

In terms of the real and imaginary parts x and y of the holomorphic coordinate $z = x + iy$, the non-trivial element of S_2 acts on $\mathbb{C}^*$ via the assignment to (x,y) of $(\frac{x}{r^2}, -\frac{y}{r^2})$ where as usual $r^2 = x^2 + y^2 = z\overline{z}$, and the two S_2-invariant functions $X = x + \frac{x}{r^2}$ and $Y = y - \frac{y}{r^2}$ on $T^{\mathbb{C}} \cong \mathbb{C}^*$ serve as real coordinate functions on the quotient $\mathbb{C}^*\big/S_2$ in such a way that the complex S_2-invariant $Z = \sigma_1 = X + iY = z + z^{-1}$ is a holomorphic coordinate. Introduce the S_2-invariant function b on $\mathbb{C}^*$ defined by

$$2b = r^2 + \frac{1}{r^2}. \tag{3.9.2}$$

This is a real algebraic function on $\mathbb{C}^*$ which is S_2-invariant and hence descends to the quotient $\mathbb{C}^*\big/S_2$ as a continuous function. Since $\sigma = X^2+Y^2-2b$, the algebra of invariants $\mathbb{R}[\mathbb{C}^*]^{S_2}$ is generated by X, Y, and b, and the relation

$$Y^2 = b^2 - 1 - (b-1)\frac{X^2+Y^2}{2}; \tag{3.9.3}$$

is equivalent to the relation (3.9.1). Moreover, on the quotient $\mathbb{C}^*\big/S_2$, the function b is non-negative whence this quotient is realized in $\mathbb{R}^3$, with coordinates X, Y, and b, as the semialgebraic set given by the equation (3.9.3) and the obvious inequality $b \geq 1$. Indeed, this inequality may also be derived from the inequality (3.6.12) and, since $\sigma = \frac{z}{\overline{z}} + \frac{\overline{z}}{z}$, the inequality $\sigma^2 \leq 4$ resulting from (3.6.13) is obviously satisfied.

As for the stratified Poisson structure, Theorem 3.8.2 or a direct calculation yields

$$\begin{aligned} \frac{i}{2}\{Z,\overline{Z}\} &= z\overline{z} + \frac{1}{z\overline{z}} - \left(\frac{z}{\overline{z}} + \frac{\overline{z}}{z}\right) \\ &= \tau - \sigma = Z\overline{Z} - 2\left(\frac{z}{\overline{z}} + \frac{\overline{z}}{z}\right) = Z\overline{Z} - 2\sigma. \end{aligned} \tag{3.9.4}$$

In terms of the real generators, the formula (3.9.4) comes down to

$$\{X,Y\} = X^2+Y^2-2\sigma = 4b - (X^2+Y^2). \tag{3.9.5}$$

Further, it is straightforward to determine the remaining Poisson brackets among the generators X, Y, and b. Hence, as a complex algebraic space, the quotient $\mathbb{C}^*\big/S_2$ is a copy of the complex line; the algebra $\mathbb{R}[\mathbb{C}^*\big/S_2]$ is the algebra of polynomial functions in the variables X, Y, b, subject to the relation (3.9.3); and the stratified symplectic Poisson bracket on this algebra is given by (3.9.5).

3.9.6. The reduced Kähler potential. Recall that the Kähler potential κ on $\mathbb{C}^*$, cf. (1.2) and (2.1.3), is given by

$$\kappa(z) = \frac{1}{4}\log^2(z\overline{z}) = \log^2(r).$$

This is a real analytic function which is manifestly S_2-invariant. Hence it descends to a continuous function κ_{red} on the quotient $\mathbb{C}^*\big/S_2$, and κ_{red} lies in $C^\omega(\mathbb{C}^*\big/S_2)$; however, even though $\mathbb{C}^*\big/S_2$ is topologically an ordinary plane, with coordinates X and Y, κ_{red} is not smooth in the usual sense at the two singular points and hence is not a smooth function of the two variables X and Y. Indeed, away from the two singular points, the function κ_{red} is an ordinary Kähler potential, but at each of these two points the Kähler structure is *not even defined* as an infinitesimal object on the *ordinary* tangent space to the quotient at each of these points, relative to the standard smooth structure on $\mathbb{C}^*\big/S_2 \cong \mathbb{C} \cong \mathbb{R}^2$. Since $Z = z + z^{-1}$, the coordinate z satisfies the quadratic equation $\left(z - \frac{Z}{2}\right)^2 = \frac{Z^2}{4} - 1$ whence, in view of (1.2), at $Z = X + iY \neq \pm 2$, the reduced potential κ_{red} is given by

$$\kappa_{\mathrm{red}}(Z) = \frac{1}{4}\log^2(z\overline{z}) = \log^2\left|\frac{Z}{2} + \sqrt{\frac{Z^2}{4} - 1}\right|. \tag{3.9.6.1}$$

Notice that, at $Z = \pm 2$, this function is manifestly not an ordinary smooth function of the variables X and Y. To justify the claim that κ_{red} lies in $C^\omega(\mathbb{C}^*\big/S_2)$ we must show that κ_{red} may be written as a real analytic function of the variables X, Y and b. In order to see this, consider the entire analytic function Ch in the variable w given by

$$\mathrm{Ch}(w) = \sum_{j=0}^{\infty} \frac{w^j}{(2j)!}.$$

For intelligibility we note that $\mathrm{Ch}(w^2) = \cosh(w)$ and $\mathrm{Ch}(-w^2) = \cos(w)$. For $w > -\pi^2$, the function Ch is strictly increasing and hence admits an analytic inverse

$$\mathrm{arCh}\colon]-1,\infty[\to]\pi^2,\infty[.$$

Recall that $t = \log r$ and that $\kappa = t^2$. Since

$$2b = r^2 + \frac{1}{r^2} = 2\cosh(2t) = 2\mathrm{Ch}(4t^2) = 2\mathrm{Ch}(4\kappa),$$

for $b \geq 1$, we have

$$\kappa = \frac{1}{4}\mathrm{arCh}(b).$$

A suitable extension of the restriction of the function arCh to $]0,\infty[$ to a smooth function defined on the entire real line yields κ as a smooth function of the variable b as desired. Thus κ_{red} may be written as a smooth function of the variable b, indeed, as a *real analytic* function of the variable b on a neighborhood of the quotient $\mathbb{C}^*\big/S_2$, realized in the copy of $\mathbb{R}^3$ with coordinates X, Y, and b.

3.9.6.2. Remark. The real Poisson structure given by (3.9.4) or (3.9.5) *cannot* be described in terms of the coordinates X and Y alone. Indeed, the (holomorphic) derivative χ' has the two fixed points 1 and -1 of the S_2-action as zeros, and hence b is not an ordinary smooth function of the variables X and Y. Solving (3.9.3) for b, and noticing that $b \geq 1$, we obtain

$$b = \frac{X^2+Y^2}{4} + \sqrt{1 + \frac{Y^2-X^2}{2} + \frac{(X^2+Y^2)^2}{16}}$$

whence, at $(X,Y) = (\pm 2, 0)$, b is not smooth as a function of the variables X and Y. Moreover, $b(1) = b(-1) = 1$ whence the Poisson bracket $\{X,Y\}$ vanishes at the two fixed points of the S_2-action.

3.9.7. Geometric Interpretation. The composite of χ with the complex exponential mapping amounts to twice the holomorphic hyperbolic cosine function. This observation provides a geometric interpretation of the adjoint quotient under discussion in terms of the geometry of hyperbolic cosine and in particular visualizes the familiar fact that, unless there is a single stratum, the strata arising from cotangent bundle reduction are *not* cotangent bundles on strata of the orbit space of the base space. The real space which underlies this quotient has received considerable attention: It is the "canoe", cf. [2], [8] (pp. 148 ff.) and [28]. This space also arises as the reduced phase space for the spherical pendulum, the two singular points being absolute equilibria. The resulting complex analytic interpretation of the spherical pendulum seems to be new, though. See [21] for details.

(3.10) $K = \mathrm{SU}(3)$, with maximal torus $T \cong S^1 \times S^1$. Complex algebraically, the map (3.8.1) for $n = 3$ comes down to the map

$$(\sigma_1, \sigma_2)\colon T^{\mathbb{C}} \longrightarrow \mathbb{C}^2, \tag{3.10.1}$$

and this map induces a complex algebraic isomorphism from the quotient of $\mathrm{SL}(3,\mathbb{C})$, realized as the orbit space $T^{\mathbb{C}}/S_3$, onto a copy $\mathbb{C}^2$ of 2-dimensional complex affine space, and we take σ_1 and σ_2 as complex coordinates on the adjoint quotient. The discussion in (3.1) and (3.7) above reveals that the complement of the top stratum amounts to the complex affine curve given by the equation

$$D_3(1, -\sigma_1, \sigma_2, -1) = 0,$$

an explicit expression for $D_3(1, -\sigma_1, \sigma_2, -1)$ being given by (3.7.1). This curve is plainly parametrized by the restriction

$$\mathbb{C}^* \to \mathbb{C} \times \mathbb{C}, \quad z \mapsto (2z + z^{-2}, z^2 + 2z^{-1}) \tag{3.10.2}$$

of (3.10.1) to the diagonal, and this holomorphic curve parametrizes the *closure* of the *middle* stratum, that is, of the stratum given by the normalized complex degree 3 polynomials with a proper double root (i. e. one which is not a triple root) and with constant coefficient equal to 1. This curve has the three (complex) singularities $(3, 3)$, $3(\eta, \eta^2)$, $3(\eta^2, \eta)$, where $\eta^3 = 1, \eta \neq 1$. These points are the images of the (conjugacy classes) of the three central elements under (3.10.1); as complex curve singularities, these singularities are cuspidal. These three points constitute the *bottom* stratum.

Since, for z with $|z| = 1$, $2\overline{z} + \overline{z}^{-2}$ equals $z^2 + 2z^{-1}$, the restriction of (3.10.2) to the *real* torus $T \subseteq T^{\mathbb{C}}$ is the parametrized real curve

$$S^1 \longrightarrow \mathbb{R}^2, \quad \mathrm{e}^{i\alpha} \mapsto (u(\alpha), v(\alpha)) \in \mathbb{R}^2, \ (\alpha \in \mathbb{R}), \tag{3.10.3}$$

where $u(\alpha) + iv(\alpha) = 2\mathrm{e}^{i\alpha} + \mathrm{e}^{-2i\alpha}$; here $\mathbb{R}^2$ is embedded into $\mathbb{C}^2$ as the real affine space of real points in $\mathbb{C}^2$ in the obvious fashion. The curve (3.10.3) is a *hypocycloid*, as noted in [7] (Section 5). The real orbit space $\mathrm{SU}(3)/\mathrm{SU}(3) \cong T/S_3$ relative to conjugation, realized within the model $T^{\mathbb{C}}/S_3$ for the categorical quotient $\mathrm{SL}(3,\mathbb{C})//\mathrm{SL}(3,\mathbb{C})$, amounts to the compact region in $\mathbb{R}^2$ enclosed by the curve (3.10.3).

As a complex algebraic stratified Kähler space, the adjoint quotient looks considerably more complicated. Indeed, the complexification $\mathbb{R}[T^{\mathbb{C}}/S_3]_{\mathbb{C}}$ of the real coordinate ring $\mathbb{R}[T^{\mathbb{C}}/S_3] \cong \mathbb{R}[T^{\mathbb{C}}]^{S_3}$ of the adjoint quotient $T^{\mathbb{C}}/S_3$, viewed as a real semialgebraic space, has the nine generators (3.7.3), subject to the relations (3.7.5), (3.7.8), and (3.7.10), together with the relations $\sigma_3 = 1$ and $\overline{\sigma}_3 = 1$. In view of the last two relations, it suffices to take the generators

$$\sigma_1, \ \overline{\sigma}_1, \ \sigma_2, \ \overline{\sigma}_2, \ \sigma, \ \rho, \ \overline{\rho}, \tag{3.10.4}$$

subject to the relations which arise from the relations (3.7.5), (3.7.8), and (3.7.10) by substitution of 1 for each occurrence of σ_3 and $\overline{\sigma}_3$. This does not change the relation

(3.7.10) since neither σ_3 nor $\overline{\sigma}_3$ occur in (3.7.10) and, applied to the relations (3.7.5) and (3.7.8), the procedure yields the relations

$$
\begin{aligned}
D_3(1,-\sigma_1,\sigma_2,-1)\overline{\sigma}_2 = {} & (9+\sigma_2^3-4\sigma_1\sigma_2)\overline{\sigma}_1^2 \\
& + (4\sigma_1^2-3\sigma_2-\sigma_1\sigma_2^2)\overline{\sigma}_1\sigma \\
& + (6\sigma_1-\sigma_2^2)\overline{\sigma}_1\rho + (\sigma_2^2-3\sigma_1)\sigma^2 \\
& + (9-\sigma_1\sigma_2)\sigma\rho + (\sigma_1^2-3\sigma_2)\rho^2
\end{aligned}
\tag{3.10.5}
$$

and

$$
\begin{aligned}
D_3(1,-\sigma_1,\sigma_2,-1) = {} & \overline{\sigma}_1^3 - \sigma_2\overline{\sigma}_1^2\sigma + (\sigma_2^2-2\sigma_1)\overline{\sigma}_1^2\rho + \sigma_1\overline{\sigma}_1\sigma^2 \\
& - ((\sigma_1^2-2\sigma_2)\sigma_1 - \sigma_1^3 + 3\sigma_1\sigma_2 - 3)\overline{\sigma}_1\sigma\rho \\
& - \sigma^3 + (\sigma_1^2-2\sigma_2)\overline{\sigma}_1\rho^2 \\
& + \sigma_2\sigma^2\rho - \sigma_1\sigma\rho^2 + \rho^3.
\end{aligned}
\tag{3.10.6}
$$

Thus the complexification $\mathbb{R}[T^{\mathbb{C}}/S_3]_{\mathbb{C}}$ of the real coordinate ring of $T^{\mathbb{C}}/S_3$ is generated by the seven bisymmetric functions (3.10.4), subject to the three relations (3.7.10), (3.10.5) and (3.10.6).

Theorem 3.8.2 yields the stratified symplectic Poisson structure on the complexification of the real coordinate ring of $T^{\mathbb{C}}/S_3$ in terms of the nine generators $\tau_{(1,0)}$, $\tau_{(2,0)}$, $\tau_{(3,0)}$, $\tau_{(0,1)}$, $\tau_{(0,2)}$, $\tau_{(0,3)}$, $\tau_{(1,1)}$, $\tau_{(1,2)}$, $\tau_{(2,1)}$. This Poisson structure has rank 4 on the top stratum, rank 2 on the middle stratum, and rank zero at the three points of the bottom stratum, that is, at the three cusps of the complex affine curve (3.10.2).

Finally, cf. (3.6) above, the reduced Kähler potential κ_{red} is the function in $C^\omega(T^{\mathbb{C}}/S_3) \cong C^\omega(T^{\mathbb{C}})^{S_3}$ which arises from the Kähler potential κ, cf. (1.2) and (2.1.3), when κ is rewritten as a real analytic function of (the real and imaginary parts of) the generators (3.10.4), similarly as in (3.9.6) above. We refrain from spelling out the details.

4. Other simple compact Lie groups

We illustrate the theory developed so far briefly for various other groups.

(4.1) B_n: $K = \mathrm{SO}(2n+1,\mathbb{R})$ $(n \geq 1)$. The adjoint quotient is isomorphic to that for the following case; indeed, the root systems B_n and C_n differs from each other only by an exchange of long and short roots. The statement of the theorem in the introduction for the group $\mathrm{SO}(2n+1,\mathbb{R})$ is a consequence of Corollary 3.4.4, combined with the fact that, under the embedding of this group into $\mathrm{U}(2n+1)$ given by the defining representation, the first n fundamental representations of $\mathrm{U}(2n+1)$ restrict to the fundamental representations of $\mathrm{SO}(2n+1,\mathbb{R})$.

(4.2) C_n: $K = \mathrm{Sp}(n)$ $(n \geq 1)$. Embed the product $\mathrm{Sp}(1) \times \cdots \times \mathrm{Sp}(1)$ of n copies of $\mathrm{Sp}(1)$ into $\mathrm{Sp}(n)$ in the standard fashion. Accordingly, the product of the standard maximal tori in these copies of $\mathrm{Sp}(1)$ yields the standard maximal torus T of $\mathrm{Sp}(n)$, and the Weyl group W amounts to the wreath product $(\mathbb{Z}/2)^n \rtimes S_n$ of $\mathbb{Z}/2$ with the symmetric group S_n on n letters. Here each copy of $Z/2$ arises as the Weyl group of a copy of $\mathrm{Sp}(1)$ and, with a grain of salt, the symmetric group S_n yields the overall

symmetries of the situation. In terms of the notation introduced in (2.2) above, we take $V=\mathbb{C}^{2n}$ and embed $T^{\mathbb{C}}$ into V in the obvious way, so that the following hold:
(i) The obvious coordinate functions $z_1,\dots,z_n,z_1^{-1},\dots,z_n^{-1}$ on $T^{\mathbb{C}}$ generating the complex algebraic coordinate ring

$$\mathbb{C}[T^{\mathbb{C}}]=\mathbb{C}[z_1,\dots,z_n,z_1^{-1},\dots,z_n^{-1}] \tag{4.2.1}$$

are independent coordinate functions on V.
(ii) The action of the j'th copy of $\mathbb{Z}/2$ in W interchanges z_j and z_j^{-1} and leaves invariant the other coordinates, and the copy of S_n in W permutes the $z_1,\dots,z_n$'s and the $z_1^{-1},\dots,z_n^{-1}$'s separately; this specifies the action of the Weyl group W.

Extending the notation introduced in (3.9) above, for $1\leq j\leq n$, let $Z_j=z_j+z_j^{-1}$. The algebra $\mathbb{C}[T^{\mathbb{C}}]^{(\mathbb{Z}/2)^n}$ of $(\mathbb{Z}/2)^n$-invariants is plainly the polynomial algebra $\mathbb{C}[Z_1,\dots,Z_n]$, and the algebra $\mathbb{C}[T^{\mathbb{C}}]^W$ of W-invariants is given by

$$\mathbb{C}[T^{\mathbb{C}}]^W=\mathbb{C}[Z_1,\dots,Z_n]^{S_n}. \tag{4.2.2}$$

This algebra of invariants is freely generated by the n elementary symmetric functions or by the first n power sum functions in the variables $Z_1,\dots,Z_n$, and the resulting Hilbert map from $T^{\mathbb{C}}$ to $\mathbb{C}^n$ induces a complex algebraic isomorphism from the adjoint quotient $T^{\mathbb{C}}\big/W$ onto a copy of $\mathbb{C}^n$. This yields the complex algebraic structure of the adjoint quotient $T^{\mathbb{C}}\big/W$. The n elementary symmetric functions in the variables $Z_1,\dots,Z_n$ yield the fundamental characters of $\mathrm{Sp}(n,\mathbb{C})$.

To derive the real structure, we note first that the complexification $\mathbb{R}[T^{\mathbb{C}}]_{\mathbb{C}}$ of the real coordinate ring $\mathbb{R}[T^{\mathbb{C}}]$ of $T^{\mathbb{C}}$ amounts to the complex algebra

$$\mathbb{R}[T^{\mathbb{C}}]_{\mathbb{C}}=\mathbb{C}[z_1,\dots,z_n,z_1^{-1},\dots,z_n^{-1},\overline{z}_1,\dots,\overline{z}_n,\overline{z}_1^{-1},\dots,\overline{z}_n^{-1}]. \tag{4.2.3}$$

Extending the notation introduced in (3.9) above further, for $1\leq j\leq n$, let

$$\overline{Z}_j=\overline{z}_j+\overline{z}_j^{-1},\ \ \sigma_{(j)}=\frac{z_j}{\overline{z}_j}+\frac{\overline{z}_j}{z_j},\ \ R_j=\sigma_{(j)}^2+Z_j^2+\overline{Z}_j^2-Z_j\overline{Z}_j\sigma_{(j)}-4.$$

In view of the discussion in (3.9) above, cf. the defining relation (3.9.1), the algebra of $(\mathbb{Z}/2)^n$-invariants is given by

$$\mathbb{R}[T^{\mathbb{C}}]_{\mathbb{C}}^{(\mathbb{Z}/2)^n}=\mathbb{C}[Z_1,\dots,Z_n,\overline{Z}_1,\dots,\overline{Z}_n,\sigma_{(1)},\dots,\sigma_{(n)}]\big/(R_1,\dots R_n). \tag{4.2.4}$$

Hence the algebra $\mathbb{R}[T^{\mathbb{C}}]_{\mathbb{C}}^W$ of W-invariants is the algebra of trisymmetric functions in the variables

$$Z_1,\dots,Z_n,\overline{Z}_1,\dots,\overline{Z}_n,\sigma_{(1)},\dots,\sigma_{(n)},$$

subject to the relations $R_1=0,\ \dots,\ R_n=0$, where the symmetric group S_n permutes the $Z_1,\dots,Z_n$'s, the $\overline{Z}_1,\dots,\overline{Z}_n$'s, and the $\sigma_{(1)},\dots,\sigma_{(n)}$'s separately in the obvious fashion. Moreover, cf. (3.9.4), the Poisson structure is induced by the identities

$$\frac{i}{2}\{Z_j,\overline{Z}_j\}=Z_j\overline{Z}_j-2\sigma_{(j)},\ \ (1\leq j\leq n). \tag{4.2.5}$$

The statement of the theorem in the introduction for the group $K = \mathrm{Sp}(n)$ is a consequence of Corollary 3.4.4, combined with the fact that, under the embedding of this group into $\mathrm{U}(2n)$ given by the defining representation, the first n fundamental representations of $\mathrm{U}(2n)$ restrict to the fundamental representations of $\mathrm{Sp}(n)$.

(4.3) D_n: $K = \mathrm{SO}(2n, \mathbb{R})$ $(n \geq 2)$. The obvious embedding of the product of n copies of $\mathrm{SO}(2, \mathbb{R})$ into $\mathrm{SO}(2n, \mathbb{R})$ yields the standard maximal torus T of $\mathrm{SO}(2n, \mathbb{R})$, and the group $(\mathbb{Z}/2)^n \rtimes S_n$ acts on T in the same fashion as on the torus in (4.2) above. However, the Weyl group $W \cong (\mathbb{Z}/2)^{n-1} \rtimes S_n$ amounts to the subgroup of $(\mathbb{Z}/2)^n \rtimes S_n$ where among the substitutions in the copy of $(\mathbb{Z}/2)^n$ only even ones are admitted.

In terms of the notation introduced in (2.2) above, similarly as in (4.2) above, we take $V = \mathbb{C}^{2n}$ and embed $T^{\mathbb{C}}$ into V in the standard way, so that the obvious coordinate functions $z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}$ on $T^{\mathbb{C}}$ generating the complex algebraic coordinate ring

$$\mathbb{C}[T^{\mathbb{C}}] = \mathbb{C}[z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}] \tag{4.3.1}$$

are independent coordinate functions on V and so that what corresponds to (4.2)(ii) still holds. As above, extending the notation introduced in (3.9), for $1 \leq j \leq n$, let $Z_j = z_j + z_j^{-1}$ and let σ_j be the j'th elementary symmetric functions in the variables $Z_1, \ldots, Z_n$. The n'th elementary symmetric function $\sigma_n(Z_1, \ldots, Z_n)$ decomposes as a sum of 2^n terms of the kind $\frac{z_1 \ldots z_k}{z_{k+1} \ldots z_n}$, and we refer to the difference $k-(n-k) = 2k-n$ as the *degree* of $\frac{z_1 \ldots z_k}{z_{k+1} \ldots z_n}$. Now $\sigma_n(Z_1, \ldots, Z_n) = Z_1 \cdot \ldots \cdot Z_n$ decomposes as a sum

$$Z_1 \cdot \ldots \cdot Z_n = \sigma_n^+(z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}) + \sigma_n^-(z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1})$$

where σ_n^+ is the sum of all terms of degrees congruent to n modulo 4 and σ_n^- the sum of all terms of degrees congruent to $n-2$ modulo 4. For example, when $n = 3$,

$$\begin{aligned} \sigma_n^+ &= z_1 z_2 z_3 + \frac{z_1}{z_2 z_3} + \frac{z_2}{z_3 z_1} + \frac{z_3}{z_1 z_2}, \\ \sigma_n^- &= \frac{z_1 z_2}{z_3} + \frac{z_2 z_3}{z_1} + \frac{z_3 z_1}{z_2} + \frac{1}{z_1 z_2 z_3} \end{aligned}$$

The functions σ_n^+ and σ_n^- are both $(\mathbb{Z}/2)^{n-1}$-invariant, even W-invariant. The n'th exterior power Λ^n of the defining representation of $\mathrm{SO}(2n+1, \mathbb{R})$, restricted to $\mathrm{SO}(2n, \mathbb{R})$ (relative to the obvious embedding), is well known to decompose as a direct sum of two irreducible $\mathrm{SO}(2n, \mathbb{R})$-representations Λ^+ and Λ^-, and σ_n^+ and σ_n^- are the characters of these representations. The algebra $\mathbb{C}[T^{\mathbb{C}}]^{(\mathbb{Z}/2)^{n-1}}$ of $(\mathbb{Z}/2)^{n-1}$-invariants is the subalgebra of (4.3.1) generated by $Z_1, \ldots, Z_n$ and σ_n^+ (or σ_n^-), and these are subject to the relation

$$(\sigma_n^+ + 2\sigma_{n-2} + \ldots)(\sigma_n^- + 2\sigma_{n-2} + \ldots) = (\sigma_{n-1} + \ldots)^2. \tag{4.3.2}$$

Consequently the complex coordinate ring $\mathbb{C}[T^{\mathbb{C}}/W]$ of the adjoint quotient $T^{\mathbb{C}}/W$ for $\mathrm{SO}(2n, \mathbb{R})$, that is, the algebra $\mathbb{C}[T^{\mathbb{C}}]^W$ of W-invariants, is the subalgebra of (4.3.1) generated by the $n-1$ elementary symmetric functions $\sigma_1, \ldots, \sigma_{n-1}$ together with σ_n^+ and σ_n^-, subject to the relation (4.3.2). That this relation is indeed defining

reflects the standard structure of the complex representation ring of $\mathrm{SO}(2n,\mathbb{R})$; it is also a consequence of the formulas (4.5.1) and (4.5.2) for δ_+^2, δ_-^2, and $\delta_+\delta_-$ in (4.5) below. For $1 \leq j \leq n-1$, when χ_j, refers to the j'th fundamental character of $\mathrm{SO}(2n,\mathbb{C})$, that is, to the character of the j'th exterior power of the definig representation, the function σ_j coincides with the character $\chi_{j-1}+\chi_j$.

The real structure and the Poisson structure can then be determined in a way similar to that explained in (4.2) above, but with an appropriate algebra of multisymmetric functions. This is a bit messy, but there is no real difficulty. We spare the reader and ourselves these added troubles here.

(4.4) B_n: $K = \mathrm{Spin}(2n+1,\mathbb{R})$ $(n \geq 2)$. The obvious embedding of the product of n copies of $\mathrm{SO}(2,\mathbb{R})$ into $\mathrm{SO}(2n+1,\mathbb{R})$ yields the standard maximal torus T of $\mathrm{SO}(2n+1,\mathbb{R})$, and the Weyl group $W \cong (\mathbb{Z}/2)^n \rtimes S_n$ acts on the maximal torus T of $\mathrm{SO}(2n+1,\mathbb{R})$ in the same fashion as on the torus in (4.2) above. We take as maximal torus $\widetilde{T}$ in $\mathrm{Spin}(2n+1,\mathbb{R})$ the pre-image of T under the canonical surjection from $\mathrm{Spin}(2n+1,\mathbb{R})$ to $\mathrm{SO}(2n+1,\mathbb{R})$, and we realize $\widetilde{T}$ as the subspace

$$\widetilde{T} = \{(z_1,\ldots,z_n,z); z_1\ldots z_n = z^2\} \subseteq T \times S^1$$

of $T \times S^1$. The action of the Weyl group W on $\widetilde{T}$, restricted to the symmetric group S_n, viewed as a subgroup of W in the obvious way, amounts simply to permutation of the $z_1,\ldots,z_n$'s while z remains fixed and, for $1 \leq j \leq n$, the unique lift to $\widetilde{T}$ of the involution which on T sends the coordinate z_j to z_j^{-1} is now given by the assignment to (z_j,z) of (z_j^{-1}, zz_j^{-1}) and leaves invariant the other coordinates. The action of the group (of order 2) of deck transformations is given by the assignment to z of $-z$ and leaves the coordinates z_1, $\ldots$, z_n invariant.

In terms of the notation introduced in (2.2) above, similarly as in (4.2) above, we take $V = \mathbb{C}^{2n+2}$ and embed $\widetilde{T}^{\mathbb{C}}$ into V in such a way that the obvious coordinate functions $z_1,\ldots,z_n,z,z_1^{-1},\ldots,z_n^{-1},z^{-1}$ on $\widetilde{T}^{\mathbb{C}}$ generating the complex algebraic coordinate ring

$$\mathbb{C}[\widetilde{T}^{\mathbb{C}}] = \mathbb{C}[z_1,\ldots,z_n,z,z_1^{-1},\ldots,z_n^{-1},z^{-1}]/(z_1\ldots z_n z^{-2}-1, z_1^{-1}\ldots z_n^{-1}z^2-1) \tag{4.4.1}$$

are independent coordinate functions on V and so that what corresponds to (4.2)(ii) still holds. As above, extending the notation introduced in (3.9), for $1 \leq j \leq n$, let $Z_j = z_j + z_j^{-1}$, and let $\sigma_1,\ldots,\sigma_n$ be the elementary symmetric functions in the variables $Z_1,\ldots,Z_n$. Moreover, let

$$\delta = z + \frac{z}{z_1} + \frac{z}{z_2} + \cdots + z^{-1} \tag{4.4.2}$$

be the $(\mathbb{Z}/2)^n$-orbit sum of z; since z is invariant under S_n, δ is invariant even under W. A calculation yields

$$\delta^2 = 2^n + 2^{n-1}\sigma_1 + \cdots + 2\sigma_{n-1} + \sigma_n. \tag{4.4.3}$$

The algebra $\mathbb{C}[\widetilde{T}^{\mathbb{C}}]^{(\mathbb{Z}/2)^n}$ of $(\mathbb{Z}/2)^n$-invariants is generated by Z_1, $\ldots$, Z_n, and δ, subject to the relation (4.4.2), and the algebra $\mathbb{C}[\widetilde{T}^{\mathbb{C}}]^W$ of W-invariants is the polynomial algebra

$$\mathbb{C}[\widetilde{T}^{\mathbb{C}}]^W = \mathbb{C}[\sigma_1,\ldots,\sigma_{n-1},\delta].$$

We note that δ is the character of the half-spin representation Δ of $\mathrm{Spin}(2n+1,\mathbb{R})$ of dimension 2^n, and the identity (4.4.3) may be rewritten as the familiar identity

$$\Delta^2 = 1 + \Lambda^1 + \dots + \Lambda^n \tag{4.4.4}$$

in the complex representation ring of $\mathrm{Spin}(2n+1,\mathbb{R})$ where as usual Λ^j $(1 \leq j \leq n)$ refers to the j'th exterior power of the ordinary vector representation of dimension $2n+1$ given by the projection to $\mathrm{SO}(2n+1,\mathbb{R})$; indeed, Λ^1 has character $\sigma_1 + 1$, Λ^2 has character $\sigma_2 + \sigma_1 + 2$, etc. The subalgebra generated by $\sigma_1, \dots, \sigma_n$ is precisely the isomorphic image in $\mathbb{C}[\widetilde{T}^{\mathbb{C}}/W] \cong \mathbb{C}[\widetilde{T}^{\mathbb{C}}]^W$ of the complex coordinate ring $\mathbb{C}[T^{\mathbb{C}}/W] \cong \mathbb{C}[T^{\mathbb{C}}]^W$ of the adjoint quotient $T^{\mathbb{C}}/W$ for $\mathrm{SO}(2n+1,\mathbb{R})$ under the canonical injection induced by the covering projection from $\widetilde{T}$ to T, the algebra $\mathbb{C}[T^{\mathbb{C}}]^W$ having been spelled out explicitly in (4.2) above. This algebra plainly coincides with the subalgebra $\mathbb{C}[\widetilde{T}^{\mathbb{C}}/W]^{\mathbb{Z}/2}$ of invariants under the induced action of the group $(\cong \mathbb{Z}/2)$ of deck transformations.

A variant of the approach in (4.2) above yields the real semialgebraic structure and the stratified Kähler structure: The complexification $\mathbb{R}[\widetilde{T}^{\mathbb{C}}]_{\mathbb{C}}$ of the real coordinate ring $\mathbb{R}[\widetilde{T}^{\mathbb{C}}]$ of $\widetilde{T}^{\mathbb{C}}$ amounts to the complex algebra

$$\mathbb{R}[\widetilde{T}^{\mathbb{C}}]_{\mathbb{C}} = \mathbb{C}[z_1, \dots, z_n, z_1^{-1}, \dots, z_n^{-1}, \overline{z}_1, \dots, \overline{z}_n, \overline{z}_1^{-1}, \dots, \overline{z}_n^{-1}, z, z^{-1}]/(S, T, \overline{S}, \overline{T}),$$

where $S = z_1 \dots z_n z^{-2} - 1$ and $T = z_1^{-1} \dots z_n^{-1} z^2 - 1$, cf. (4.4.1). Let $\sigma_{(n+1)}$ be the $(\mathbb{Z}/2)^n$-orbit sum of $\frac{z}{\overline{z}}$. With the notation introduced above, the algebra $\mathbb{R}[\widetilde{T}^{\mathbb{C}}]_{\mathbb{C}}^{(\mathbb{Z}/2)^n}$ of $(\mathbb{Z}/2)^n$-invariants is generated by

$$Z_1, \dots, Z_n, \delta, \overline{Z}_1, \dots, \overline{Z}_n, \overline{\delta}, \sigma_{(1)}, \dots, \sigma_{(n)}, \sigma_{(n+1)}, \{\delta, \overline{\delta}\},$$

subject to suitable relations, and the induced S_n-action permutes the $Z_1, \dots, Z_n$'s, the $\overline{Z}_1, \dots, \overline{Z}_n$'s, and the $\sigma_{(1)}, \dots, \sigma_{(n)}$'s separately and leaves δ, $\overline{\delta}$, $\sigma_{(n+1)}$, and $\{\delta, \overline{\delta}\}$ invariant. We shall comment on the Poisson bracket $\{\delta, \overline{\delta}\}$ below. Hence the algebra $\mathbb{R}[\widetilde{T}^{\mathbb{C}}]_{\mathbb{C}}^W$ of W-invariants arises from the algebra of trisymmetric functions in the variables

$$Z_1, \dots, Z_n, \overline{Z}_1, \dots, \overline{Z}_n, \sigma_{(1)}, \dots, \sigma_{(n)}, \tag{4.4.5}$$

together with the four invariants δ, $\overline{\delta}$, $\{\delta, \overline{\delta}\}$ and $\sigma_{(n+1)}$.

We will now explain how the stratified Poisson structure can be determined. The Poisson structure on the complexification $\mathbb{R}[\widetilde{T}^{\mathbb{C}}]_{\mathbb{C}}$ of the real coordinate ring of $\widetilde{T}^{\mathbb{C}}$ is given by the formulas

$$\frac{i}{2}\{z_j, \overline{z}_j\} = 2 z_j \overline{z}_j \ \ (1 \leq j \leq n), \tag{4.4.6}$$

$$\frac{i}{2}\{z, \overline{z}\} = z \overline{z}, \tag{4.4.7}$$

$$\frac{i}{2}\{z_j, \overline{z}\} = z_j \overline{z} \ \ (1 \leq j \leq n), \tag{4.4.8}$$

similar to the formulas (4.2.5). The factor 2 in (4.4.6) has been introduced for convenience, in particular to arrive at simple formulas; without the factor 2 in (4.4.6) we would need the factor $\frac{1}{2}$ in (4.4.7) and (4.4.8). This fixes of course the Kähler potential κ on $K^{\mathbb{C}}$. A tedious but straightforward calculation yields uniquely determined W-invariants A_n and B_n such that

$$\frac{i}{2}\{\delta,\overline{\delta}\} = A_n - 2\sigma_{(n+1)} \tag{4.4.9}$$

and

$$\delta\overline{\delta} = A_n + B_n. \tag{4.4.10}$$

For example, when $n = 1$ so that $\mathrm{Spin}(3,\mathbb{R}) \cong \mathrm{SU}(2)$, $B_1 = 0$,

$$\delta\overline{\delta} = A_1 = (z + z^{-1})(\overline{z} + \overline{z}^{-1}),$$

and (4.4.9) comes down to (3.9.4). Likewise, when $n = 2$,

$$A_2 = (z + z^{-1})(\overline{z} + \overline{z}^{-1}) + \left(\frac{z}{z_1} + \frac{z}{z_2}\right)\left(\frac{\overline{z}}{\overline{z}_1} + \frac{\overline{z}}{\overline{z}_2}\right).$$

It is now straightforward to complete the determination of the stratified Kähler structure on the adjoint quotient under discussion. We do not pursue this here.

This discussion reveals that the statement of the theorem in the introduction is not true for the group $K = \mathrm{Spin}(2n+1,\mathbb{R})$ when $n \geq 2$. Indeed, as a module over the complexification of the real coordinate ring of the adjoint quotient for $\mathrm{SO}(2n+1,\mathbb{R})$, the algebra of W-invariants $\mathbb{R}[\widetilde{T}^{\mathbb{C}}]^W_{\mathbb{C}}$ is generated by δ, $\overline{\delta}$, $\{\delta,\overline{\delta}\}$ and $\sigma_{(n+1)}$, but *not* by δ, $\overline{\delta}$, $\{\delta,\overline{\delta}\}$ alone.

(4.5) D_n: $K = \mathrm{Spin}(2n,\mathbb{R})$ ($n \geq 2$). Relative to the standard embedding of $\mathrm{Spin}(2n,\mathbb{R})$ into $\mathrm{Spin}(2n+1,\mathbb{R})$, the maximal torus $\widetilde{T}$ in $\mathrm{Spin}(2n+1,\mathbb{R})$ is a maximal torus in $\mathrm{Spin}(2n,\mathbb{R})$. As in (4.3) above, the Weyl group $W \cong (\mathbb{Z}/2)^{n-1} \rtimes S_n$ amounts to the subgroup of $(\mathbb{Z}/2)^n \rtimes S_n$ where among the substitutions in the copy of $(\mathbb{Z}/2)^n$ only even ones are admitted. The invariant δ introduced in (4.4) above decomposes as a sum $\delta = \delta^+ + \delta^-$ where δ^+ and δ^- are the characters of the positive and negative half-spin representations Δ^+ and Δ^-, respectively, of $\mathrm{Spin}(2n,\mathbb{R})$ of dimension 2^{n-1}, and the algebra $\mathbb{C}[\widetilde{T}^{\mathbb{C}}]^W$ of W-invariants is the polynomial algebra

$$\mathbb{C}[\widetilde{T}^{\mathbb{C}}]^W = \mathbb{C}[\sigma_1,\ldots,\sigma_{n-2},\delta^+,\delta^-].$$

Moreover, calculations yield the familiar identities in the representation ring of $\mathrm{Spin}(2n,\mathbb{R})$ relating the products $\delta^+\delta^+$, $\delta^+\delta^-$, $\delta^-\delta^-$ with the elementary symmetric functions $\sigma_1,\ldots,\sigma_{n-1}$ and the functions σ_n^+ and σ_n^-. For n odd,

$$\begin{aligned} \delta_+^2 &= \sigma_+ + 2\sigma_{n-2} + \cdots + 2^{n-2}\sigma_1 \\ \delta_-^2 &= \sigma_- + 2\sigma_{n-2} + \cdots + 2^{n-2}\sigma_1 \\ \delta_+\delta_- &= \sigma_{n-1} + 2\sigma_{n-3} + \cdots + 2^{n-1} \end{aligned} \tag{4.5.1}$$

while for n even,

$$\begin{aligned} \delta_+^2 &= \sigma_+ + 2\sigma_{n-2} + \cdots + 2^{n-1} \\ \delta_-^2 &= \sigma_- + 2\sigma_{n-2} + \cdots + 2^{n-1} \\ \delta_+\delta_- &= \sigma_{n-1} + 2\sigma_{n-3} + \cdots + 2^{n-2}\sigma_1 \end{aligned} \tag{4.5.2}$$

The non-trivial involution in the group ($\cong \mathbb{Z}/2$) of deck transformations sends δ^+ and δ^- to $-\delta^+$ and $-\delta^-$, respectively, and the subalgebra $\mathbb{C}[\widetilde{T}^{\mathbb{C}}/W]^{\mathbb{Z}/2}$ of invariants under the induced action of this group is the subalgebra generated by $\sigma_1, \ldots, \sigma_{n-1}, \sigma_n^+, \sigma_n^-$. This algebra is precisely the algebra $\mathbb{C}[T^{\mathbb{C}}]^W$ spelled out in (4.3) above; it manifestly coincides with the isomorphic image in $\mathbb{C}[\widetilde{T}^{\mathbb{C}}/W] \cong \mathbb{C}[\widetilde{T}^{\mathbb{C}}]^W$ of the complex coordinate ring $\mathbb{C}[T^{\mathbb{C}}/W] \cong \mathbb{C}[T^{\mathbb{C}}]^W$ of the adjoint quotient $T^{\mathbb{C}}/W$ for $\mathrm{SO}(2n,\mathbb{R})$ under the canonical injection induced by the covering projection from $\widetilde{T}$ to T.

The stratified Kähler structure can then be determined explicitly in a way similar to that hinted at in (4.4) above. We do not pursue this here. We also note that the statement of the theorem in the introduction is not true for $K = \mathrm{Spin}(2n,\mathbb{R})$ when $n \geq 4$.

(4.6) $K = G_{2(-14)}$. This is the group of automorphisms of the octonions $\mathbb{O}$. The defining representation has real dimension 7; it is the subspace $\mathbb{O}_0$ of trace zero octonions. The real dimension of $G_{2(-14)}$ equals 14. The long roots constitute the root system A_2 and, accordingly, $G_{2(-14)}$ contains a copy of SU(3) which, in turn, admits an obvious interpretation in terms of the geometry of the octonions. The Weyl group W being a dihedral group ($\cong \mathbb{Z}/6 \rtimes \mathbb{Z}/2$) of order 12, decomposes as $W \cong W_{\mathrm{long}} \rtimes W_{\mathrm{short}}$, where $W_{\mathrm{long}} \cong S_3$ is the Weyl group of the subsystem of long roots and where $W_{\mathrm{short}} \cong \mathbb{Z}/2$ may be taken as being generated by the reflection in the short simple root. The standard maximal torus

$$T = \{\mathrm{diag}(z_1, z_2, z_3);\ z_1z_2z_3 = 1\}$$

for SU(3) is a maximal torus of $G_{2(-14)}$ as well and, in terms of this torus, the Weyl group is generated by the Weyl group $W_{\mathrm{long}} \cong S_3$ of SU(3) together with the involution which sends (z_1, z_2, z_3) to $(z_1^{-1}, z_2^{-1}, z_3^{-1})$. Indeed, this involution realizes the obvious outer automorphism of SU(3) which corresponds to the obvious symmetry of the root system A_2 interchanging the two roots. With the notation $\sigma_1 = z_1+z_2+z_3$ and $\sigma_2 = z_1z_2 + z_1z_3 + z_2z_3$ used earlier, the complex coordinate ring $\mathbb{C}[T^{\mathbb{C}}/S_3]$ of the adjoint quotient for SU(3) is the complex polynomial algebra generated by σ_1 and σ_2, and these are the characters of the defining representation (say) $V_3 \cong \mathbb{C}^3$ of $\mathrm{SL}(3,\mathbb{C})$ and of the second exterior square $\Lambda^2_{\mathbb{C}}V_3 \cong \overline{V}_3$ of the defining representation, respectively. Indeed, the outer automorphism of SU(3) corresponding to the symmetry of the root system interchanges the two fundamental representations and passes to the involution on the adjoint quotient $T^{\mathbb{C}}/S_3 \cong \mathbb{C}^2$ for SU(3) which interchanges the complex coordinate functions σ_1 and σ_2. Hence the complex coordinate ring of the adjoint quotient $T^{\mathbb{C}}/W$ for $G_{2(-14)}$ is the algebra

$$\mathbb{C}[T^{\mathbb{C}}/W] = \mathbb{C}[\sigma_1, \sigma_2]^{\mathbb{Z}/2} = \mathbb{C}[\Sigma_1, \Sigma_2]$$

where $\Sigma_1 = \sigma_1 + \sigma_2$ and $\Sigma_2 = \sigma_1\sigma_2$. In particular, complex algebraically, the adjoint quotient $T^{\mathbb{C}}/W$ for $G_{2(-14)}$ is again a copy of $\mathbb{C}^2$.

To interpret this description in terms of the two fundamental characters of $G_{2(-14)}$, we note that, as a real SU(3)-representation, the defining representation of $G_{2(-14)}$ decomposes as $V_3 \oplus \mathbb{R}$ where V_3 is viewed as a real 6-dimensional representation of SU(3). Hence the defining representation has as character the function $\Sigma_1 + 1$. The other fundamental representation of $G_{2(-14)}$ is the adjoint representation. Since the tensor product $V_3 \otimes \Lambda^2 V_3 \cong V_3 \otimes \overline{V}_3$ of the two fundamental representations of SU(3) decomposes as the direct sum $\mathfrak{su}(3) \oplus \mathbb{R}$ of the adjoint representation and the trivial 1-dimensional representation, the adjoint representation of SU(3) has character $\sigma_1\sigma_2 - 1$. As a real SU(3)-representation, the adjoint representation of $G_{2(-14)}$ decomposes as $\mathfrak{su}(3) \oplus V_3$. Hence the adjoint representation of $G_{2(-14)}$ has as character the function $\Sigma_1 + \Sigma_2 - 1$. We note that, as a complex SU(3)-representation, the adjoint representation of $G_{2(-14)}$ decomposes as $\mathfrak{sl}(3,\mathbb{C}) \oplus V_3 \oplus \overline{V}_3$.

The real semialgebraic structure and the stratified Kähler structure arise from that spelled out in (3.10) above by taking $(\mathbb{Z}/2)$-invariants. In view of the discussion in (3.10) above, the complexification $\mathbb{R}[T^{\mathbb{C}}/W_{\mathrm{long}}]_{\mathbb{C}}$ of the real coordinate ring of the adjoint quotient $T^{\mathbb{C}}/W_{\mathrm{long}}$ for SU(3) is generated by the seven invariants σ_1, $\overline{\sigma}_1$, σ_2, $\overline{\sigma}_2$, σ, ρ, $\overline{\rho}$, cf. (3.10.4), subject to the three relations (3.7.10), (3.10.5) and (3.10.6). The non-trivial involution coming from the induced action of $W_{\mathrm{short}} \cong \mathbb{Z}/2$ on $\mathbb{R}[T^{\mathbb{C}}/W_{\mathrm{long}}]_{\mathbb{C}}$ interchanges, respectively, σ_1, $\overline{\sigma}_1$, ρ and σ_2, $\overline{\sigma}_2$, $\overline{\rho}$, and sends σ to a W_{long}-invariant function (say) $\overline{\sigma}$ on $T^{\mathbb{C}}$ which, on the real torus T, viewed as a subspace of $T^{\mathbb{C}}$, coincides with σ. The real semialgebraic structure and the Poisson structure can then be determined in a way similar to that explained in (4.2) above, but with an appropriate algebra of quatrisymmetric functions, relative to the group $W_{\mathrm{short}} \cong \mathbb{Z}/2$.

Similarly as before, the statement of the theorem in the introduction for the group $K = G_{2(-14)}$ is a consequence of Corollary 3.4.4, combined with the observation that, under the embedding of $G_{2(-14)}$ into U(7) given by the defining representation $\mathbb{O}_0$ of $G_{2(-14)}$, (i) *the first fundamental* (i. e. defining) *representation of* U(7) *restricts to the first fundamental* (i. e. defining) *representation of* $G_{2(-14)}$, and (ii) *the second fundamental representation of* U(7) (the second exterior square of the defining representation of U(7)) *restricts to the sum of the two fundamental complex representations* $\mathfrak{g}_2(\mathbb{C})$ *and* $\mathbb{O}_0 \otimes \mathbb{C}$ *of* $G_{2(-14)}$. Indeed, (i) is obvious. To justify (ii), we note first that the defining representation $\mathbb{O}_0$ of $G_{2(-14)}$ is real and embeds $G_{2(-14)}$ into $SO(7,\mathbb{R})$ and, as a $G_{2(-14)}$-representation, the adjoint representation (Lie algebra) $\mathfrak{so}(7,\mathbb{R})$ decomposes as

$$\mathfrak{so}(7,\mathbb{R}) \cong \mathfrak{g}_{2(-14)} \oplus \mathbb{O}_0.$$

The adjoint representation of $SO(7,\mathbb{R})$ is the second exterior square of the defining representation of $SO(7,\mathbb{R})$ whence (ii).

(4.7) $K = F_{4(-52)}$. This is the group of automorphisms of the exceptional Jordan algebra $\mathcal{H}_3(\mathbb{O})$ of hermitian (3×3)-matrices over the octonions $\mathbb{O}$. The defining representation has real dimension 26; it is the subspace of trace zero matrices in $\mathcal{H}_3(\mathbb{O})$. The defining representation may also be obtained from a reductive decomposition of $\mathfrak{e}_{6(-78)}$ of the kind $\mathfrak{e}_{6(-78)} = \mathfrak{f}_4 + \mathfrak{p}$. The real dimension of $F_{4(-52)}$ equals 52. The

long roots constitute the root system D_4 and, accordingly, $F_{4(-52)}$ contains a copy of $\mathrm{Spin}(8,\mathbb{R})$ which, in turn, admits an obvious interpretation in terms of the geometry of the octonions: Denote by V_8 the (real) vector representation and by S_+ and S_- the (real) positive and negative spinor representations, respectively, of $\mathrm{Spin}(8,\mathbb{R})$. The division algebra structure on $\mathbb{O}$ corresponds to a *triality* $V_8 \oplus S_+ \oplus S_- \to \mathbb{R}$, and $\mathrm{Spin}(8,\mathbb{R})$ is the symmetry group of this triality, cf. [1]. The subgroup of $\mathrm{Spin}(8,\mathbb{R})$ which consists of the symmetries of the triality which are automorphism of $\mathbb{O}$ is precisely the group $G_{2(-14)}$. The Weyl group of W of $F_{4(-52)}$ decomposes as $((\mathbb{Z}/3)^3 \rtimes \mathbb{S}_4) \rtimes S_3$ where the copy of $(\mathbb{Z}/3)^3 \rtimes \mathbb{S}_4$ is the Weyl group W_{long} of the system of long roots, that is, of $\mathrm{Spin}(8,\mathbb{R})$. The maximal torus

$$\widetilde{T} \cong \{(z_1,z_2,z_3,z_4,z);\, z_1z_2z_3z_4 = z^2\} \subseteq (S^1)^5$$

for $\mathrm{Spin}(8,\mathbb{R})$ spelled out in (4.5) above for the general case of $\mathrm{Spin}(2n,\mathbb{R})$ is a maximal torus of $F_{4(-52)}$ as well and, in terms of this torus, the Weyl group is generated by the Weyl group of $\mathrm{Spin}(8,\mathbb{R})$ together with certain involutions; these involutions come from the obvious outer automorphisms of $\mathrm{Spin}(8,\mathbb{R})$ which correspond to the obvious symmetry group S_3 of the root system D_4 permuting the three boundary roots. This symmetry group was discovered already by E. Cartan. It is generated by the reflections in the short simple roots, and we will denote it by W_{short}. The vector representation and the positive and negative spinor representations of $\mathrm{Spin}(8,\mathbb{R})$ are all isomorphic under these outer automorphisms of $\mathrm{Spin}(8,\mathbb{R})$.

With the notation σ_1, σ_2 for the first two elementary symmetric functions in the variables Z_1,Z_2,Z_3,Z_4 and the notation δ^+, and δ^- introduced in (4.5) above, the complex coordinate ring $\mathbb{C}[T^{\mathbb{C}}/W_{\mathrm{long}}]$ of the adjoint quotient for $\mathrm{Spin}(8,\mathbb{R})$ is the complex polynomial algebra generated by σ_1, σ_2, δ^+, and δ^-. Here δ^+ and δ^- are the characters of the positive and negative spinor representations and, when χ_1 refers to the character of the vector representation V_8 of $\mathrm{Spin}(8,\mathbb{R})$ and χ_2 to that of the adjoint representation, σ_1 coincides with χ_1+1 and σ_2 with $\chi_1+\chi_2$.

The W_{short}-action passes to an S_3-action on the adjoint quotient $T^{\mathbb{C}}\big/W_{\mathrm{long}} \cong \mathbb{C}^4$ for $\mathrm{Spin}(8,\mathbb{R})$ which permutes the complex coordinate functions χ_1, δ^+, and δ^-, whence the complex coordinate ring of the adjoint quotient $T^{\mathbb{C}}\big/W$ for $F_{4(-52)}$ is the algebra

$$\mathbb{C}[T^{\mathbb{C}}/W] = \mathbb{C}[\chi_1,\chi_2,\delta^+,\delta^-]^{S_3} = \mathbb{C}[\Sigma_1,\chi_2,\Sigma_2,\Sigma_3],$$

where Σ_1, Σ_2, and Σ_3 are the elementary symmetric functions in the variables χ_1,δ^+,δ^-. Since, as a $\mathrm{Spin}(8,\mathbb{R})$-representation,

$$\mathcal{H}_3(\mathbb{O}) \cong \mathbb{R}^3 \oplus V_8 \oplus S_+ \oplus S_-,$$

the character of the defining representation of $F_{4(-52)}$ is the function Σ_1+2. Furthermore, since, as a $\mathrm{Spin}(8,\mathbb{R})$-representation,

$$\mathfrak{f}_{4(-52)} \cong \mathfrak{so}(8,\mathbb{R}) \oplus V_8 \oplus S_+ \oplus S_-,$$

the sum

$$\sigma_2 + \delta = \Sigma_1 + \chi_2$$

is the character of the adjoint representation of $F_{4(-52)}$. This is the second fundamental representation of $F_{4(-52)}$. We do not make explicit the relationship between the virtual characters $\Sigma_1, \chi_2, \Sigma_2, \Sigma_3$ and the two remaining fundamental representations of $F_{4(-52)}$.

In particular, complex algebraically, the adjoint quotient $T^{\mathbb{C}}\big/W$ for $F_{4(-52)}$ is the quotient of the adjoint quotient $T^{\mathbb{C}}\big/W_{\mathrm{long}} \cong \mathbb{C}^4$ for $\mathrm{Spin}(8,\mathbb{R})$ relative to the induced S_3-action, and this quotient is again a copy of $\mathbb{C}^4$. Moreover, the real semialgebraic structure arises from that hinted at in (4.5) in the special case where $n=4$ by the operation of taking S_3-invariants and, likewise, the stratified Kähler structure can be determined explicitly by an application of the procedure of taking S_3-invariants to the stratified Kähler structure on the adjoint quotient $T^{\mathbb{C}}\big/W_{\mathrm{long}} \cong \mathbb{C}^4$ for $\mathrm{Spin}(8,\mathbb{R})$ which, in turn, was hinted at in (4.5) above. We do not pursue this here.

5. Energy quantization on the adjoint quotient

Choose a *dominant* Weyl chamber in the Lie algebra $\mathfrak{t}$ of the maximal torus T of K and let R^+ be the resulting system of positive roots. Let Δ_K denote the *Casimir* operator on K associated with the bi-invariant Riemannian metric on K, and let $m = \dim K$. When $X_1, \dots, X_m$ is an orthonormal basis of $\mathfrak{k}$,

$$\Delta_K = X_1^2 + \cdots + X_m^2$$

in the universal algebra $\mathrm{U}(\mathfrak{k})$ of $\mathfrak{k}$, cf. e. g. [31] (p. 591). The Casimir operator depends only on the Riemannian metric, though. Since the metric on K is bi-invariant, so is the operator Δ_K; hence, by Schur's lemma, for each highest weight λ, the isotypical $(K\times K)$-summand $L^2(K,dx)_\lambda \subseteq L^2(K,dx)$ associated with λ in the Peter-Weyl decomposition of $L^2(K,dx)$ is an eigenspace, whence the representative functions are eigenfunctions for Δ_K. Let $\rho = \frac{1}{2}\sum_{\alpha\in R^+}\alpha$, so that 2ρ is the sum of the positive roots. The eigenvalue of Δ_K corresponding to the highest weight λ is known to be given explicitly by $-\varepsilon$ where

$$\varepsilon_\lambda = (|\lambda+\rho|^2 - |\rho|^2),$$

cf. e. g. [14] (Ch. V.1 (16) p. 502). The present sign is dictated by the interpretation in terms of the energy spelled out below. Thus Δ_K acts on each isotypical summand $L^2(K,dx)_\lambda$ as scalar multiplication by $-\varepsilon_\lambda$. The Casimir operator is known to coincide with the nonpositive Laplace-Beltrami operator associated with the (bi-invariant) Riemannian metric on K, see e. g. [41] (A 1.2). In the Schrödinger picture (vertical quantization on T^*K), the operator $\widehat{E}_K$ which arises as the unique extension of $-\frac{1}{2}\Delta_K$ to an unbounded self-adjoint operator on $L^2(K,dx)$ is the quantum mechanical *energy* operator associated with the Riemannian metric, whence the spectral decomposition of this operator refines in the standard manner to the Peter-Weyl decomposition of $L^2(K,dx)$ into isotypical $(K\times K)$-summands. The energy operator may by obtained by vertical quantization of the geodesic flow.

Let ε be the symplectic volume form on $\mathrm{T}^*K \cong K^{\mathbb{C}}$ (inducing Liouville measure). Define the function $\eta\colon K^{\mathbb{C}} \longrightarrow \mathbb{R}$ by

$$\eta(x,Y) = \left(\det\left(\frac{\sin(\mathrm{ad}(Y))}{\mathrm{ad}(Y)}\right)\right)^{\frac{1}{2}}, \ \ x\in K,\ Y\in\mathfrak{k}; \tag{5.1}$$

this yields a non-negative real analytic function on $K^{\mathbb{C}}$ which depends only on the variable $Y \in \mathfrak{k}$ and, for $x \in K$ and $Y \in \mathfrak{k}$, we will also write $\eta(Y)$ instead of $\eta(x,Y)$. The function η^2 is the density of Haar measure relative to the Liouville volume measure on $K^{\mathbb{C}}$, see e. g. [12] (Lemma 5). Both measures are K-bi-invariant; in particular, as a function on $\mathfrak{k}$, η is $\mathrm{Ad}(K)$-invariant.

Half-form Kähler quantization on $\mathrm{T}^*K \cong K^{\mathbb{C}}$ is accomplished by the Hilbert space $\mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa}\eta\varepsilon)$ of holomorphic functions which are square integrable relative to the measure $\mathrm{e}^{-\kappa}\eta\varepsilon$ [13], [22]. Via the embedding of $\mathfrak{k}$ into $\mathfrak{k}^{\mathbb{C}}$, the operator Δ_K is a differential operator on $K^{\mathbb{C}}$. In view of Theorem 5.2 in [22] which, in turn, is a consequence of the holomorphic Peter-Weyl theorem [22], in the holomorphic quantization on $\mathrm{T}^*K \cong K^{\mathbb{C}}$, the unique extension $\widehat{E}_{K^{\mathbb{C}}}$ of $-\frac{1}{2}\Delta_K$ to an (unbounded) self-adjoint operator on $\mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa}\eta\varepsilon)$ is the quantum mechanical *energy* operator associated with the Riemannian metric, and the spectral decomposition of this operator refines to the holomorphic Peter-Weyl decomposition of $\mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa}\eta\varepsilon)$ into isotypical $(K \times K)$-summands; see [22] (Sections 6 and 7) for details.

As before let $T^{\mathbb{C}} \subseteq K^{\mathbb{C}}$ be the complexification of T and W the Weyl group. We will now explain how *half-form Kähler quantization of the reduced kinetic energy* $\frac{1}{2}\kappa_{\mathrm{red}}$ *on the adjoint quotient of* $K^{\mathbb{C}}$ *leads to the the irreducible characters of* $K^{\mathbb{C}}$. To this end, let dt be Haar measure on $T^{\mathbb{C}}$, and let $d[t]$ be the measure on $T^{\mathbb{C}}/W$ which, multiplied by the order $|W|$ of the Weyl group W, is the push forward of the measure dt on $T^{\mathbb{C}}$ under the projection from $T^{\mathbb{C}}$ to $T^{\mathbb{C}}/W$. The Haar measure dt actually coincides with Liouville measure on $T^{\mathbb{C}}$ and, on the regular part of $T^{\mathbb{C}}\big/W$, viewed as an ordinary smooth symplectic manifold, $d[t]$ coincides with Liouville measure; we will therefore write $\varepsilon_{\mathrm{red}}$ for $d[t]$ as well. Given a highest weight λ for K, let $\chi_\lambda \colon K^{\mathbb{C}} \to \mathbb{C}$ be the irreducible (algebraic) character of $K^{\mathbb{C}}$ with highest weight λ. Each such character χ_λ of $K^{\mathbb{C}}$ manifestly passes to an algebraic function on the adjoint quotient $T^{\mathbb{C}}\big/W$ and we denote this function by $[\chi_\lambda]$.

The *saturation* of the zero locus $\mu^{-1}(0)$ is the $K^{\mathbb{C}}$-closure $K^{\mathbb{C}}\mu^{-1}(0) \subseteq K^{\mathbb{C}}$ of $\mu^{-1}(0)$ in $K^{\mathbb{C}}$ relative to the (conjugation) action of $K^{\mathbb{C}}$ on itself, and the inclusion $\mu^{-1}(0) \subseteq K^{\mathbb{C}}\mu^{-1}(0)$ induces a homeomorphism from the reduced space $(\mathrm{T}^*K)_0 = \mu^{-1}(0)\big/K$ onto the $K^{\mathbb{C}}$-quotient $K^{\mathbb{C}}\mu^{-1}(0)\big/K^{\mathbb{C}}$. This yields an alternate description of the quotient $K^{\mathbb{C}}//K^{\mathbb{C}} \cong T^{\mathbb{C}}/W$ of $K^{\mathbb{C}}$. Relative to the projection to the quotient, consider the push forward to $T^{\mathbb{C}}/W$ of the measure $\mathrm{e}^{-\kappa}\eta\varepsilon = \frac{\mathrm{e}^{-\kappa}}{\eta}dxdY$ on $K^{\mathbb{C}}$. This push forward measure on $T^{\mathbb{C}}/W$ has a density relative to Liouville measure $\varepsilon_{\mathrm{red}}$ and hence can be written in the form

$$\mathrm{e}^{-\kappa_{\mathrm{red}}}\gamma\varepsilon_{\mathrm{red}}, \tag{5.2}$$

for a uniquely determined real valued function γ on $T^{\mathbb{C}}\big/W$ such that, given two holomorphic K-invariant functions Φ and Ψ on $K^{\mathbb{C}}$ that are square integrable relative to the measure $\mathrm{e}^{-\kappa}\eta\varepsilon$, when Φ_{red} and Ψ_{red} denote the induced holomorphic functions on the quotient $T^{\mathbb{C}}\big/W$,

$$\int_{K^{\mathbb{C}}} \overline{\Phi}\Psi\mathrm{e}^{-\kappa}\eta\varepsilon = \int_{T^{\mathbb{C}}/W} \overline{\Phi_{\mathrm{red}}}\Psi_{\mathrm{red}}\mathrm{e}^{-\kappa_{\mathrm{red}}}\gamma\varepsilon_{\mathrm{red}}. \tag{5.3}$$

To establish the existence of the function γ consider the conjugation mapping

$$q^{\mathbb{C}} \colon (K^{\mathbb{C}}/T^{\mathbb{C}}) \times T^{\mathbb{C}} \longrightarrow K^{\mathbb{C}},\ (yT^{\mathbb{C}}, t) \mapsto yty^{-1},\ y \in K^{\mathbb{C}}, t \in T^{\mathbb{C}},$$

and integrate the induced $(2m)$-form $(q^{\mathbb{C}})^*(\mathrm{e}^{-\kappa}\eta\varepsilon)$ over "the fiber" $K^{\mathbb{C}}/T^{\mathbb{C}}$. In view of the Gaussian constituent $\mathrm{e}^{-\kappa}$, this integration is a well defined operation. Let $\widetilde{\gamma}$ be the density of the resulting $(2n)$-form on $T^{\mathbb{C}}$ relative to the Liouville volume form on $T^{\mathbb{C}} \cong \mathrm{T}^*T$ where $n = \dim T$ and let $\widehat{\gamma} = \widetilde{\gamma}/|W|$ where $|W|$ denotes the order of the Weyl group W. The function $\widehat{\gamma}$ descends to a function on the quotient $T^{\mathbb{C}}/W$, and dividing this function by $\mathrm{e}^{-\kappa_{\mathrm{red}}}$ we obtain the function γ we are looking for.

The half-form quantization procedure on $K^{\mathbb{C}}$ induces a half-form quantization procedure on the adjoint quotient $T^{\mathbb{C}}/W$; we do not spell out the details here. The above discussion then leads to the following result, which includes the quantization of the reduced kinetic energy $\frac{1}{2}\kappa_{\mathrm{red}} \in C^{\omega}(T^{\mathbb{C}}/W)$.

Theorem 5.4. *The quantum Hilbert space for the stratified Kähler structure on the adjoint quotient $T^{\mathbb{C}}/W$ amounts to the Hilbert space $\mathcal{H}L^2(T^{\mathbb{C}}/W, \mathrm{e}^{-\kappa_{\mathrm{red}}}\gamma\varepsilon_{\mathrm{red}})$ of holomorphic functions on $T^{\mathbb{C}}/W$ that are square-integrable with respect to the measure $\mathrm{e}^{-\kappa_{\mathrm{red}}}\gamma\varepsilon_{\mathrm{red}}$, and this Hilbert space is freely spanned by the holomorphic functions $[\chi_\lambda]$ on $T^{\mathbb{C}}/W$ that correspond to the irreducible characters of $K^{\mathbb{C}}$. Furthermore, the reduced energy operator $\widehat{E}_{\mathrm{red}}$ is given by*

$$\widehat{E}_{\mathrm{red}}[\chi_\lambda] = \varepsilon_\lambda[\chi_\lambda].$$

Indeed, let ε_T be the Liouville volume form on $T^{\mathbb{C}} \cong \mathrm{T}^*T$. The restriction mapping induces a unitary isomorphism

$$\mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa}\eta\varepsilon)^K \longrightarrow \mathcal{H}L^2(T^{\mathbb{C}}, \mathrm{e}^{-\kappa}\widehat{\gamma}\varepsilon_T)^W$$

of Hilbert spaces, and the canonical map

$$\mathcal{H}L^2(T^{\mathbb{C}}/W, \mathrm{e}^{-\kappa_{\mathrm{red}}}\gamma\varepsilon_{\mathrm{red}}) \longrightarrow \mathcal{H}L^2(T^{\mathbb{C}}, \mathrm{e}^{-\kappa}\widehat{\gamma}\varepsilon_T)^W$$

is an isomorphism of Hilbert spaces. We shall give elsewhere an intrinsic description of the quantization downstairs on $\mathcal{H}L^2(T^{\mathbb{C}}/W, \mathrm{e}^{-\kappa_{\mathrm{red}}}\gamma\varepsilon_{\mathrm{red}})$. The advantage of the description in terms of the quotient $T^{\mathbb{C}}/W$ rather than in terms of the W-invariants of $\mathcal{H}L^2(T^{\mathbb{C}}, \mathrm{e}^{-\kappa}\widetilde{\gamma}\varepsilon_T)$ is that it brings the *costratified* nature of the Hilbert space structure to the fore:

The holomorphic quantization procedure in [19] yields a *costratified Hilbert space*, that is, a system of Hilbert spaces, one Hilbert space for the closure of each stratum, with bounded linear operators among these Hilbert spaces which correspond to the closure relations among the strata. Such a system is *structure on the quantum level which has the classical singularities as its shadow*. Under the present circumstances, the costratified structure arises in the following fashion: As explained before, the adjoint quotient $T^{\mathbb{C}}/W$ is decomposed into strata, and the closure of a stratum is an affine complex variety. Given a stratum S, let I_S be the ideal of functions in the complex coordinate ring $\mathbb{C}[T^{\mathbb{C}}/W]$ which vanish on S or, equivalently, on its closure $\overline{S}$, let $\widehat{I}_S \subseteq \mathcal{H}L^2(T^{\mathbb{C}}/W, \mathrm{e}^{-\kappa_{\mathrm{red}}}\gamma\varepsilon_{\mathrm{red}})$ be the closed subspace spanned by I_S, and let $\mathcal{H}_S$ be the orthogonal complement of $\widehat{I}_S$ in $\mathcal{H}L^2(T^{\mathbb{C}}/W, \mathrm{e}^{-\kappa_{\mathrm{red}}}\gamma\varepsilon_{\mathrm{red}})$. Equivalently, we can characterize $\widehat{I}_S$ as the space of holomorphic functions in $\mathcal{H}L^2(T^{\mathbb{C}}/W, \mathrm{e}^{-\kappa_{\mathrm{red}}}\gamma\varepsilon_{\mathrm{red}})$ that vanish on S and, as a complex vector space, $\mathcal{H}_S$ is the space of holomorphic

functions on the complex manifold S which arise as restrictions of holomorphic functions on $T^{\mathbb{C}}/W$ which are square-integrable relative to the measure $\mathrm{e}^{-\kappa_{\mathrm{red}}}\gamma\varepsilon_{\mathrm{red}}$. In particular, when S is the top stratum, $\mathcal{H}_S$ coincides with the entire Hilbert space $\mathcal{H}L^2(T^{\mathbb{C}}/W, \mathrm{e}^{-\kappa_{\mathrm{red}}}\gamma\varepsilon_{\mathrm{red}})$. Thus, for each stratum S, there is a canonical projection from $\mathcal{H}L^2(T^{\mathbb{C}}/W, \mathrm{e}^{-\kappa_{\mathrm{red}}}\gamma\varepsilon_{\mathrm{red}})$ to $\mathcal{H}_S$ and, given two strata S_1 and S_2 with $S_2 \subseteq \overline{S_1}$, there is a unique projection $\mathcal{H}_{S_1} \to \mathcal{H}_{S_2}$. The system $\{\mathcal{H}_S\}$, as S ranges over the strata, together with the projections $\mathcal{H}_{S_1} \to \mathcal{H}_{S_2}$ whenever $S_2 \subseteq \overline{S_1}$, constitutes a *costratified* Hilbert space. Alternatively, the quantum Hilbert space is spanned by the functions $[\chi_\lambda] \in \mathbb{C}[T^{\mathbb{C}}/W]$ induced by the irreducible characters as λ ranges over the highest weights. The closure of a stratum is a complex affine variety in $T^{\mathbb{C}}/W$ ($\cong \mathbb{C}^n$ when K is semisimple and simply connected of rank n), and the Hilbert space corresponding to that stratum is simply the quotient of $\mathcal{H}L^2(T^{\mathbb{C}}/W, \mathrm{e}^{-\kappa_{\mathrm{red}}}\gamma\varepsilon_{\mathrm{red}})$ obtained when the functions $[\chi_\lambda]$ are restricted to that stratum. In [23], the quantum mechanics on such a costratified Hilbert space is worked out in detail for the special case where $K = \mathrm{SU}(2)$.

Thus the measure giving rise to the Hilbert space $\mathcal{H}L^2(T^{\mathbb{C}}/W, \mathrm{e}^{-\kappa_{\mathrm{red}}}\gamma\varepsilon_{\mathrm{red}})$ on the reduced level, viz. the measure $\mathrm{e}^{-\kappa_{\mathrm{red}}}\gamma\varepsilon_{\mathrm{red}}$, appears to be formally of the same kind as the measure $\mathrm{e}^{-\kappa}\eta\varepsilon$ which determines the unreduced Hilbert space $\mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa}\eta\varepsilon)$, that is to say, on the reduced level, the measure involves the reduced energy κ_{red}, the reduced Liouville volume form $\varepsilon_{\mathrm{red}}$, and a certain correction term γ. However, there is a fundamental difference: On the unreduced level, the correction term η comes from the metaplectic correction and is intrinsically defined in terms of the geometry of the group $K^{\mathbb{C}}$ whereas the correction term γ on the reduced level is not merely defined in terms of the geometry of the adjoint quotient $K^{\mathbb{C}}/\!/K^{\mathbb{C}} \cong T^{\mathbb{C}}/W$ and, in a sense, encapsulates part of the history as to how the quotient arises. Relative to these measures, in the case at hand, *half-form Kähler quantization commutes with reduction, observables and Hilbert space structures included*, the reduced space being endowed with a measure which is not in an obvious way related with merely the geometry of the reduced space.

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